# Math 220A – Lecture 7

October 23, 2023

# I. Last time

## Cauchy's Theorem$^{+}$ (Homotopy version).

$(+)$ $f : U \longrightarrow \mathbb{C}$ continuous, holomorphic in $U \setminus \{a\}$

$\Longrightarrow \displaystyle\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$ if $\gamma_0 \overset{U}{\sim} \gamma_1$ are piecewise $C^1$ loops.

We needed this stronger form to prove Homotopy Cauchy Integral formula:

last time

## Cauchy's Integral Formula (CIF)

$f : U \longrightarrow \mathbb{C}$ holomorphic, $\gamma \overset{U}{\sim} 0$, $a \in U \setminus \{\gamma\}$

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a}\, dz$$

# Proof of Homotopy Cauchy$^+$ ↙ Conway IV.6.7.

Recall the assumption

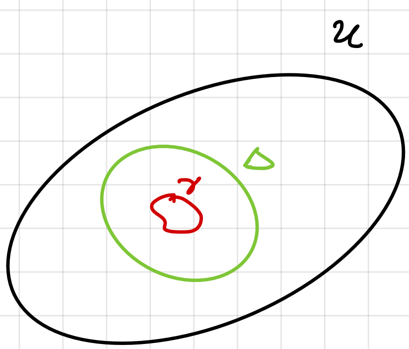
$(+)$ $f : U \longrightarrow \mathbb{C}$ continuous, holomorphic in $U \setminus \{a\}$

For the proof we only use

$\boxed{i}$ $f$ continuous

$\boxed{ii}$ $\forall\ \Delta \subseteq U$, $\{\gamma\} \subseteq \Delta$ piecewise $C^1$ loop

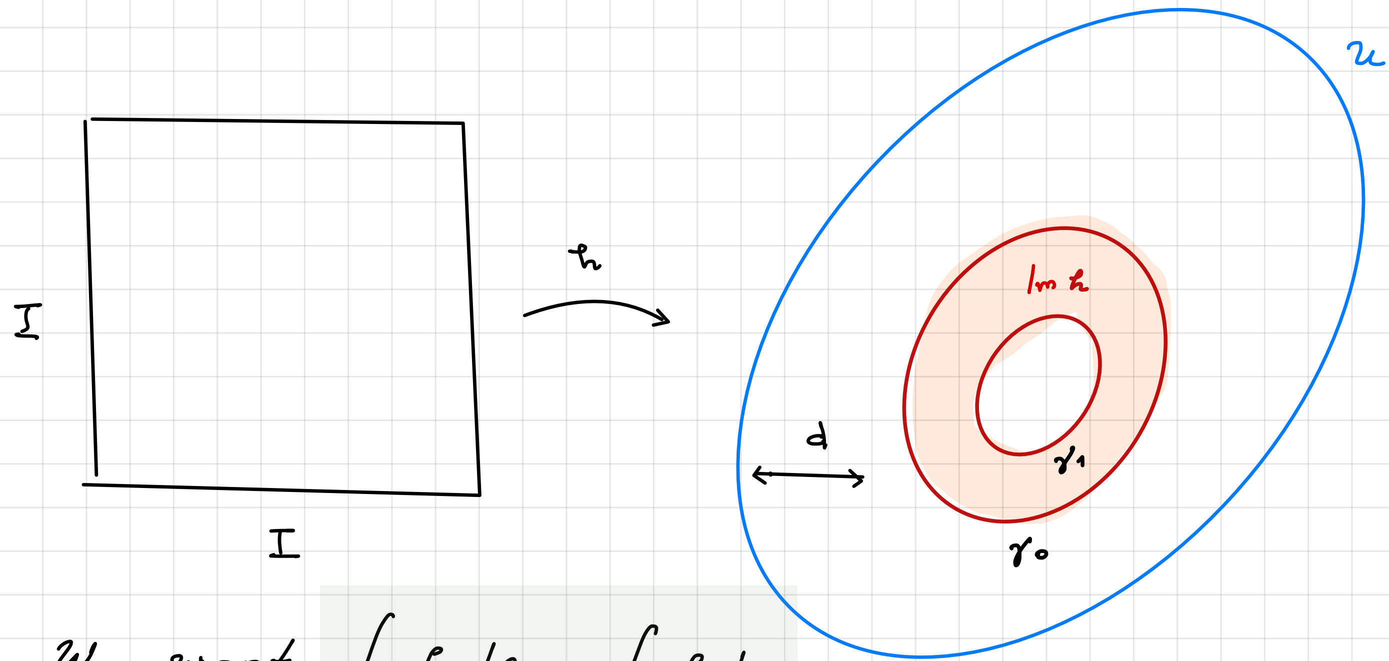
$$\Longrightarrow \int_\gamma f\, dz = 0 \qquad (*)$$



We have seen $(+)$ implies $(*)$ in Lecture 5, Corollary$^+$, stated below:

Corollary$^+$ $f : \Delta \longrightarrow \mathbb{C}$ continuous, holomorphic in $\Delta \setminus \{a\}$

$$\Longrightarrow \int_\gamma f\, dz = 0 \quad \forall\ \gamma \text{ piecewise } C^1 \text{ loop}$$




We want $\boxed{\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.}$

Let $h : I \times I \longrightarrow \mathcal{U}$ be the homotopy from $\gamma_0$ to $\gamma_1$

$\operatorname{Im} h$ compact, $\mathbb{C} \setminus \mathcal{U}$ closed $\Longrightarrow \exists\, d > 0$.

$$\boxed{d = \operatorname{dist}(\operatorname{Im} h, \mathbb{C} \setminus \mathcal{U})} \qquad (1)$$

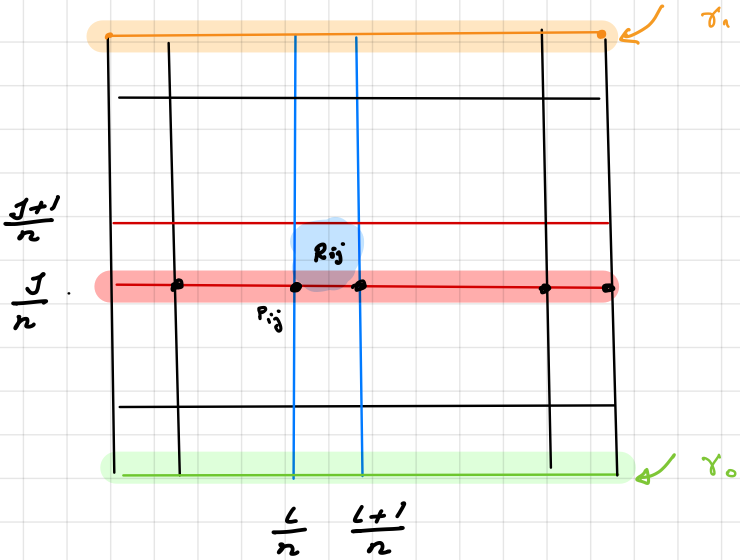
$h$ continuous, $I \times I$ compact $\Longrightarrow$ $h$ uniformly continuous

$\Longrightarrow \exists\, \delta > 0$ such that

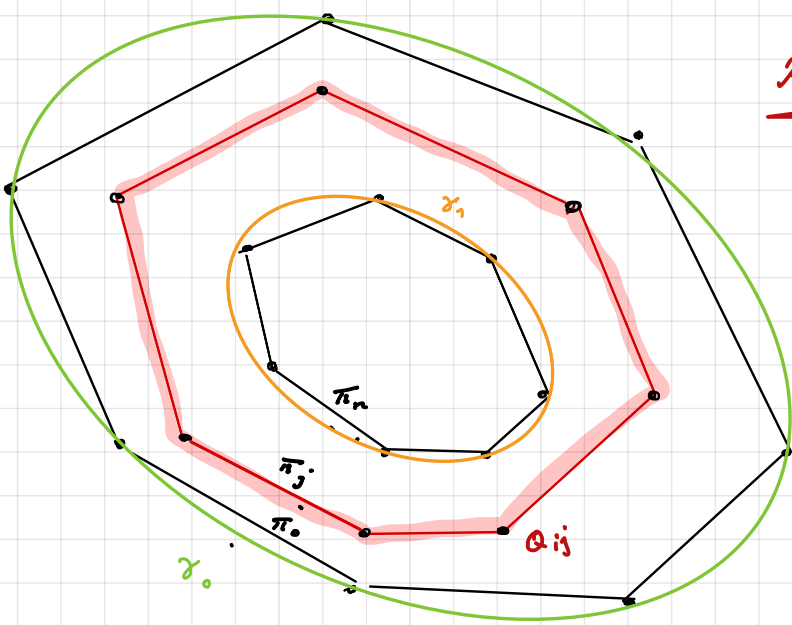
$$\boxed{|s - s'| < \delta,\ |t - t'| < \delta \Rightarrow |h(s,t) - h(s',t')| < d} \qquad (2)$$

Let $\frac{1}{n} < \delta$ for $n \in \mathbb{Z}_+$. Subdivide $I$ into $n$ equal

$\left[\frac{i}{n}, \frac{i+1}{n}\right]$ of length $\frac{1}{n} < \delta$.




- Let $P_{ij}$ have coordinates $\left(\frac{i}{n}, \frac{j}{n}\right)$.
- Let $Q_{ij} = h(P_{ij}) \in \mathrm{Im}\, h$
- Let $R_{ij}$ be the rectangle $\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$
- Let $\Delta_{ij} = \Delta(Q_{ij}, d)$.

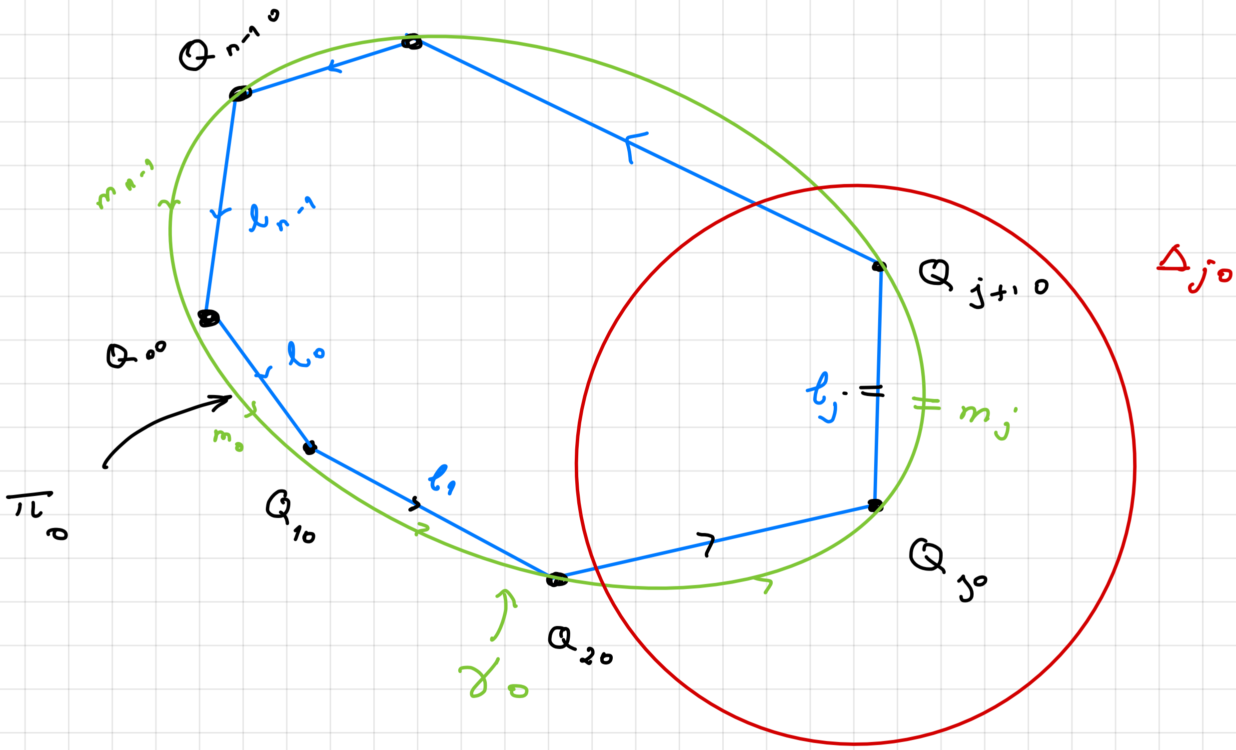

## Remarks

$\boxed{I}$ $\Delta_{ij} \subseteq \mathcal{U}$ because of the choice of $d$ in (1)

$\boxed{II}$ $h(R_{ij}) \subseteq \Delta_{ij}$ because the sides of $R_{ij}$ have length $\frac{1}{n} < \delta$ and uniform continuity (2).

Let $\pi_j$ be the polygon $Q_{0j}, Q_{1j}, \dots Q_{nj} = Q_{0j}$.

Claim $\boxed{a}$ $\quad \int_{\gamma_0} f\,dz = \int_{\pi_0} f\,dz$ & $\int_{\pi_n} f\,dz = \int_{\gamma_1} f\,dz.$




Let $l_0, l_1 \dots l_{n-1}$ be the edges of the polygon $\pi_0$.

$m_0, m_1, \dots, m_{n-1}$ be the arcs of the curve $\gamma_0$.

i.e. $\quad m_j = \gamma_0 \big|_{[\frac{j}{n}, \frac{j+1}{n}]}$

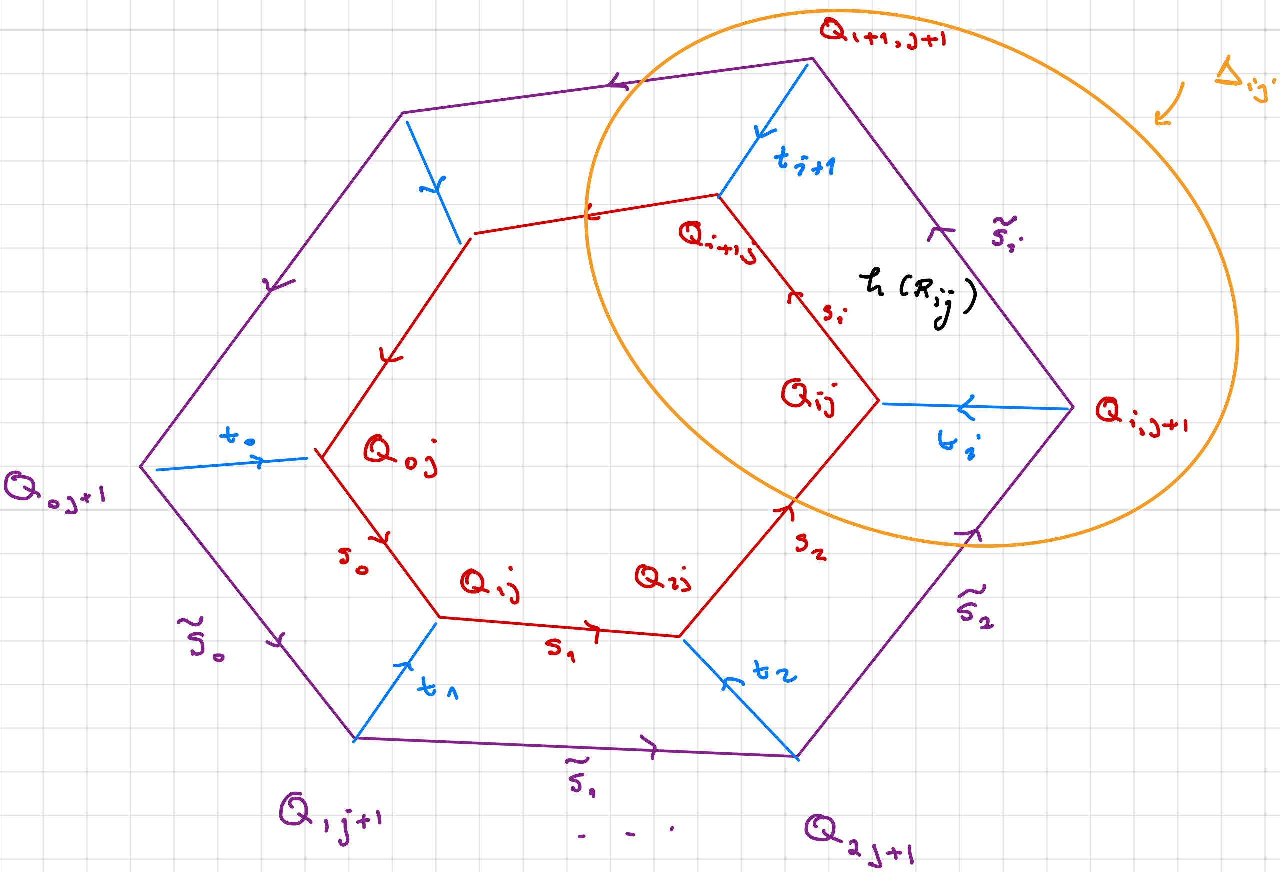
Since $Q_{j+1,0} \underset{\boxed{III}}{\in} \Delta_{j0} \underset{\boxed{II}}{\subseteq} \mathcal{U}$, we must have $l_j, m_j$ are in $\Delta_{j0}$.

By (*) we have $\displaystyle\int_{l_j + (-m_j)} f\,dz = 0 \implies \int_{l_j} f\,dz = \int_{m_j} f\,dz$

Adding for all $j$, we find $\displaystyle\int_{\pi_0} f\,dz = \int_{\gamma_0} f\,dz.$

## Claim $\boxed{6}$

$$\int_{\pi_j} f\,dz = \int_{\pi_{j+1}} f\,dz$$




Let $s_0, \dots s_{n-1}$ be the edges of $\pi_j$.

$\tilde{s}_0, \dots \tilde{s}_{n-1}$ the edges of $\pi_{j+1}$

$t_0, \dots, t_{n-1}$ the segments joining $Q_{ij+1}$ to $Q_{ij}$. $t_i$

Since $h(R_{ij}) \subseteq \Delta_{ij} \Rightarrow \tilde{s}_i + t_{i+1} + (-s_i) + (-t_i)$ is a loop in $\Delta_{ij}$. $\rightarrow$ by $\boxed{1c}$ above

By (*) we find

$$\int_{\tilde{s}_i + t_{i+1} + (-s_i) + (-t_i)} f \, dz = 0$$

$$\Longrightarrow \int_{\tilde{s}_i} f \, dz - \int_{s_i} f \, dz = \int_{t_i} f \, dz - \int_{t_{i+1}} f \, dz$$

Add these for all $i$. We find

$$\int_{\pi_j} f \, dz - \int_{\pi_{j+1}} f \, dz = 0 \Rightarrow \text{Claim } \boxed{b}.$$

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From Claims $\boxed{a}$ & $\boxed{b}$,

$$\int_{\gamma_0} f \, dz = \int_{\pi_0} f \, dz = \dots = \int_{\pi_n} f \, dz = \int_{\gamma_1} f \, dz.$$

Q.E.D.

## II. Taylor Expansion ← Conway IV. 2. 8

**Theorem** $f : U \longrightarrow \mathbb{C}$ holomorphic, $a \in U$, $\Delta(a, R) \subseteq U$.

Then in $\Delta(a, R)$, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k \qquad (*).$$

for some $a_k \in \mathbb{C}$.

$\Longrightarrow$ $f$ analytic $\Longrightarrow$ $f$ is $\infty$ - many times differentiable.

ANALYTIC = HOLOMORPHIC = DIFFERENTIABLE

**Proof** Let $\Delta(a, R) \subseteq U$. We pick $0 < r < R$. Let $z \in \Delta(a, r)$. By CIF

$$f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t - z} \, dt$$

**Key**: $$\frac{1}{t - z} = \frac{1}{t - a - (z - a)} = \frac{1}{t-a} \cdot \frac{1}{1 - \frac{z-a}{t-a}}$$

$$= \frac{1}{t-a} \sum_{k=0}^{\infty} \frac{(z-a)^k}{(t-a)^k} \quad \text{converges since} \quad \left|\frac{z-a}{t-a}\right| = \frac{|z-a|}{r} < 1.$$

$$\Longrightarrow \frac{f(t)}{t-z} = \sum_{k=0}^{\infty} f(t) \cdot \frac{(z-a)^k}{(t-a)^{k+1}} . \quad (+)$$

Claim $(+)$ converges uniformly in $t$ over $|t-a| = r$.

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Indeed, let $f_k(t) = f(t) \cdot \frac{(z-a)^k}{(t-a)^{k+1}}$. Let $|f(t)| \le M$

for $|t-a| = r \Longrightarrow |f_k(t)| \le M \cdot \frac{|z-a|^k}{r^{k+1}} = M_k$

Note $\sum M_k < \infty$ since $|z-a| < r$. Thus the claim follows

by Weierstraß M-test

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Since the convergence is uniform, we can integrate

$$\Longrightarrow f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t-z} \, dt \overset{(+)}{=}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}} \, dt \cdot (z-a)^k$$

$$= \sum_{k=0}^{\infty} a_k (z-a)^k .$$

Note that

$$a_k = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt$$ does not depend on $r$.

Indeed, this follows by Homotopy Cauchy

$$\int_{\gamma_r} \frac{f(t)}{(t-a)^{k+1}}\, dt = \int_{\gamma_{r'}} \frac{f(t)}{(t-a)^{k+1}}\, dt$$

since $\underbrace{\gamma_r}_{\{|t-a|=r\}} \sim \underbrace{\gamma_{r'}}_{\{|t-a|=r'\}}$ in a region $V = \{ r-\varepsilon < |z-a| < r'+\varepsilon \} \subseteq \mathcal{U}$

& $\dfrac{f(t)}{(t-a)^{k+1}}$ holomorphic in $V$

**Remark** In the proof, $z$ was fixed and we showed

$$f(z) = \sum_k a_k (z-a)^k$$ pointwise in $\Delta(a,R)$.

However, after the fact, we must also have local uniform convergence using the results on power series in Lecture 2.

**Remark** $f: U \longrightarrow \mathbb{C}$, $\overline{\Delta}(a, r) \subseteq U$.

$$a_k = \frac{f^{(k)}(a)}{k!} = \underbrace{\frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt}_{\text{from the proof of the Theorem.}}$$

Lecture 2

$\Downarrow$

Conway IV. 2.13

**Theorem** $f: U \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(a, r) \subseteq U$. Then

$$f^{(k)}(a) = \frac{k!}{2\pi i} \underbrace{\int_{|t-a|=r}}_{\text{positively oriented}} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

We will show that a similar formula holds at all points of a disc, not only the center.

# Cauchy's Estimates ~ Conway IV. 2.14

Let $f: U \to \mathbb{C}$ holomorphic, $\overline{\Delta}(a, R) \subseteq U$. Let

$$M_R = \sup_{|z-a|=R} |f(z)|.$$

Then $|f^{(k)}(a)| \leq k! \dfrac{M_R}{R^k}$.

**Remark** $k = 0$:

$$|f(a)| \leq \sup_{z \in \partial\Delta(a,R)} |f(z)|$$

**Proof** By CIF for derivatives

$$|f^{(k)}(a)| = \left| \frac{k!}{2\pi i} \int_{|z-a|=R} \frac{f(z)}{(z-a)^{k+1}} \, dz \right|$$

$$\leq \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot \text{length } |z-a|=R$$

$$= \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot 2\pi R = k! \frac{M_R}{R^k}.$$