Math 220A – Lecture 8

October 25, 2023

Last time

Conway IV.2.13

Theorem $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(a,r) \subseteq \mathcal{U}$. Then

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

$|t-a|=r$ positively oriented

Cauchy's Estimates $\rightsquigarrow$ Conway IV.2.14

Let $f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $\overline{\Delta}(a,R) \subseteq \mathcal{U}$. Let

$$M_R = \sup_{|z-a|=R} |f(z)|.$$

Then $|f^{(k)}(a)| \leq k! \dfrac{M_R}{R^k}$.

Improve the theorem?

We next show that a similar formula holds at all points of a disc, not only the center. & also for more general loops.

Homotopy Cauchy Integral Formula for derivatives

$f : U \longrightarrow \mathbb{C}$ holomorphic, $\gamma \overset{U}{\sim} 0$, $a \in U \setminus \{\gamma\}$.

$$n(\gamma, a)\, f^{(k)}(a) = \frac{k!}{2\pi i} \int_\gamma \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

Proof Use Homotopy Cauchy Integral Formula & differentiate under the integral sign. (Problem 5, HWK 3).

In more detail,

$$n(\gamma, a)\, f(a) \overset{\text{Cauchy}}{=} \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t-a}\, dt$$

$$\overset{\text{HWK3}}{\Longrightarrow} \quad n(\gamma, a)\, f^{(k)}(a) = \frac{1}{2\pi i} \int_\gamma f(t) \cdot \frac{\partial^k}{\partial^k a} \cdot \left(\frac{1}{t-a} \right) dt$$

$$= \frac{k!}{2\pi i} \int_\gamma \frac{f(t)}{(t-a)^{k+1}}\, dt$$

Remark As a special case, we obtain

Theorem If $\overline{\Delta} \subseteq U$, $a \in \Delta$, $f: U \longrightarrow \mathbb{C}$ holomorphic, then

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial \Delta} \frac{f(t)}{(t-a)^{k+1}}\, dt.$$

possibly not the center of Δ

Example $$\int_{|z|=r} \frac{e^{z}}{(z-a)^{k}}\, dz \quad , r \neq |a|$$

- If $|a| > r$ the answer is 0 because the integrand is holomorphic

- If $r > |a|$, apply CIF for derivatives:

$$\frac{1}{(k-1)!} \cdot 2\pi i \cdot \partial^{(k-1)} e^{z} \Big|_{z=a} = \frac{e^{a}}{(k-1)!} \cdot 2\pi i$$

2! Entire functions

Definition A holomorphic $f: \mathbb{C} \longrightarrow \mathbb{C}$ is said to be entire.

Remark f entire $\overset{\text{Taylor}}{\Longrightarrow} f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbb{C}$

Example e^z, $\sin z$, $\cos z$ are entire.

Liouville's theorem ↙ Conway IV. 3.4

If $f: \mathbb{C} \longrightarrow \mathbb{C}$ entire & bounded $\Longrightarrow f$ constant.

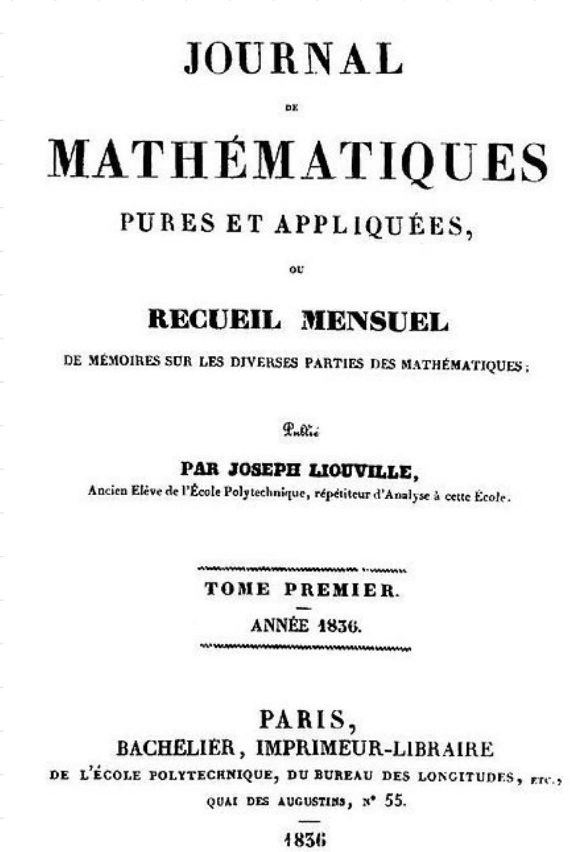
Proof: Assume $|f(z)| \leq M \quad \forall z \in \mathbb{C}$.

Cauchy's estimate for $k=1$. Take $\bar{\Delta}(a, R) \subseteq \mathbb{C}$.

$$|f'(a)| \leq \frac{M_R}{R} \leq \frac{M}{R}.$$

Take $R \longrightarrow \infty$.

Thus $f'(a) = 0$. $\forall a \Rightarrow f$ constant.



JOURNAL

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PURES ET APPLIQUÉES,

OU

RECUEIL MENSUEL

DE MÉMOIRES SUR LES DIVERSES PARTIES DES MATHÉMATIQUES;

Publié

PAR JOSEPH LIOUVILLE,

Ancien Elève de l'École Polytechnique, répétiteur d'Analyse à cette École.

TOME PREMIER.

ANNÉE 1836.

PARIS,

BACHELIER, IMPRIMEUR-LIBRAIRE

DE L'ÉCOLE POLYTECHNIQUE, DU BUREAU DES LONGITUDES, ETC.,

QUAI DES AUGUSTINS, N° 55.

1836

Joseph Liouville

1809 – 1882

Journal de Liouville

Known for: Liouville's theorem

Sturm-Liouville theory

Liouville numbers

Liouville function

...

Remark $\sin z$, $\cos z$ are not bounded in $\mathbb{C}$.

Indeed,

$$\cos(\pi i n) = \frac{e^{\pi n} + e^{-\pi n}}{2} \longrightarrow \infty, \text{ as } n \longrightarrow \infty.$$

Fundamental Theorem of Algebra

Conway III. 3. 5

Any nonconstant polynomial $f \in \mathbb{C}[z]$ has at least one complex root.

Proof: WLOG f monic

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n.$$

Assume

f has no roots $\Rightarrow$ $f(z) \neq 0 \quad \forall z$.

Let $g = \frac{1}{f}$. $\Rightarrow$ g is entire. We show g bounded $\Rightarrow$

$\Rightarrow$ g constant. $\Rightarrow$ f constant. This is a contradiction.

We show g bounded. If $|z| = R$

$$|f(z)| = \left| z^n + a_1 z^{n-1} + \dots + a_n \right| \geq |z|^n - \sum_{k=1}^{n-1} |a_k| \, |z|^{n-k}$$

$$= R^n - \sum_{k=1}^{n-1} |a_k| \, R^{n-k} \longrightarrow \infty \quad \text{as} \quad R \longrightarrow \infty.$$

If $R \geq R_0 \Rightarrow |f(z)| \geq 1. \Rightarrow |g(z)| \leq 1.$

If $R \leq R_0 \Rightarrow$ by continuity of g: $|g(z)| \leq K.$

$\Rightarrow$

$\rightarrow |g(z)| \leq M = \max(1, K). \quad \forall z$, as claimed.

3 Zeros of holomorphic functions — Conway IV. 3.

$f: \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, $f \not\equiv 0$, $\mathcal{U}$ open + connected.

Def $a \in \mathcal{U}$ is a zero of order N if

$$f(a) = 0, \quad f'(a) = 0, \quad \dots, \quad f^{(N-1)}(a) = 0, \quad f^{(N)}(a) \neq 0$$

$\Rightarrow$ Taylor expansion in $\Delta(a, R) \subseteq \mathcal{U}$

$$f(z) = \sum_{k \geq N} \frac{f^{(k)}(a)}{k!} \cdot (z-a)^k = (z-a)^N g(z) \qquad (*)$$

where g is a power series converging in $\Delta(a, R)$.

$$g(a) = \frac{f^{(N)}(a)}{N!} \neq 0.$$

We need to rule out the case $N = \infty$.

Conway IV. 3. 7.

Lemma $f : U \longrightarrow \mathbb{C}$, U connected. TFAE

[i] $f \equiv 0$

[ii] $\exists\, a \in U$, $f^{(k)}(a) = 0 \quad \forall k$

[iii] $S = \{ z : f(z) = 0 \}$ has a limit point in U.

Proof [i] $\Rightarrow$ [ii], [i] $\Rightarrow$ [iii] are clear.

[iii] $\Rightarrow$ [ii] Let a be a limit point for S, $a \in U$.

Clearly $f(a) = 0$. Let us assume a has finite order N.

By $(*)$, $f(z) = (z-a)^N g(z)$ in $\Delta(a, R)$. with

g power series, $g(a) \neq 0$. By continuity of g, $g(z) \neq 0$ in

some $\Delta(a, r) \subseteq \Delta(a, R)$. Then

$$S \cap \Delta(a, r) = \{ z : (z-a)^N g(z) = 0 \} = \{a\}.$$

contradiction with a being a limit point.

Thus $N = \infty$. $\Rightarrow$ [ii].

$\boxed{ii} \Rightarrow \boxed{i}$. Let $A = \{a : f^{(k)}(a) = 0 \ \forall k\} \subseteq U$.

By assumption $A \neq \emptyset$. We show A is closed & open.

Thus $A = U$. $\Rightarrow f \equiv 0$.

- A closed. Indeed $A = \bigcap_{k=0}^{\infty} \left(f^{(k)}\right)^{-1}(0) =$ closed.

Since $f^{(k)}$ is continuous $\Rightarrow f^{(k)^{-1}}(0)$ is closed $\Rightarrow$ A closed

- A open. Let $a \in A$. By Taylor if $\Delta(a, R) \subseteq U$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = 0 \quad \text{since } f^{(k)}(a) = 0.$$

Since $f = 0$ in $\Delta(a, R) \Rightarrow f^{(k)} = 0$ in $\Delta(a, R) \Rightarrow$

$\Rightarrow \Delta(a, R) \subseteq A \Rightarrow A$ open.

<u>Identity Principle</u> $\swarrow$ Conway IV. 3. 8.

If $f, g : U \longrightarrow \mathbb{C}$ holomorphic, U connected, and

$S = \{z : f(z) = g(z)\}$ has a limit point in $U \Rightarrow f = g$.

<u>Proof</u> Let $h = f - g$. Apply the previous lemma.

Remarks

$\boxed{\text{i}}$ The zeros of $f: U \longrightarrow \mathbb{C}$ holomorphic cannot have a limit point in U.

Let $f(z) = \sin \frac{1+z}{1-z}$ holomorphic in $\mathbb{C} \setminus \{1\} := U$

Zeros $\quad \frac{1+z}{1-z} = n\pi \iff z = \frac{-1+n\pi}{1+n\pi} \longrightarrow 1.$

Thus the zeros can accumulate to ∂U.

$\boxed{\text{ii}}$ This fails for C^∞- functions

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-\frac{1}{x^2}} \sin \frac{1}{x}, & x \neq 0. \end{cases}$$

Check f is C^∞. Also f has zeros at $\frac{1}{n\pi} \longrightarrow 0.$

which has a limit point.

$\boxed{\text{iii}}$ If $f, g : U \longrightarrow \mathbb{C}$, and $\exists\, V \subseteq U$ open with $f = g$ in V, then $f \equiv g$ in U.

connected.

$\boxed{\text{IV}}$ $f \not\equiv 0$ has at most countably many zeros in U.

Let $U = \bigcup_{n=1}^{\infty} K_n$ where K_n compact. In each compact set K_n, f can only have finitely many zeros.

(indeed this is because zero(f) can't accumulate in K_n)

$$\Rightarrow \quad \text{Zero}(f) = \bigcup_n \Big(\underbrace{\text{Zero}(f) \cap K_n}_{\text{finite}} \Big) = \text{countable.}$$

Aufgaben und Lehrsätze,

erstere aufzulösen, letztere zu beweisen.

1.

(Von Herrn *N. H. Abel.*)

49. Theorème. Si la somme de la série infinie

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_m x^m + \ldots$$

est égale à zéro pour toutes les valeurs de x entre deux limites réelles α et β; on aura nécessairement

$$a_0 = 0, \; a_1 = 0, \; a_2 = 0 \ldots a_m = 0 \ldots$$

en vertu de ce que la somme de la série s'évanouira pour une valeur quelconque de x.

Identity theorem: Crelle's Journal 1827, page 286

Main theorems

$\boxed{i}$ Identity Principle – see above

$\boxed{ii}$ Open Mapping Theorem (OMT)

$\boxed{iii}$ Maximum Modulus Principle (MMP)

Recall: $f: X \longrightarrow Y$ is open map if $\forall\, U \subseteq X$ open, $f(U)$ is open.

- $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longrightarrow x^2$ is not open, $U = (-1,1)$, $f(U) = [0,1)$
- $f: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longrightarrow z^2$ is open. This is because:

Conway IV. 7.5

Theorem (Open Mapping Thm).

$f: U \longrightarrow \mathbb{C}$ not constant holomorphic $\Rightarrow f$ is open.

Theorem (Maximum Modulus) Conway IV. 3.11

$f: U \longrightarrow \mathbb{C}$ holomorphic, non constant. then $|f|$ cannot have a local maximum in U.

Remark If $\mathcal{U}$ bounded, $f : \overline{\mathcal{U}} \longrightarrow \mathbb{C}$ continuous, f holomorphic in $\mathcal{U}$, then

$$\max_{\overline{\mathcal{U}}} |f| = \max_{\partial \mathcal{U}} |f|$$

Indeed, $\overline{\mathcal{U}}$ and $\partial \mathcal{U}$ are closed & bounded $\Rightarrow$ compact. Thus $|f|$ admits maxima over $\overline{\mathcal{U}}$ & $\partial \mathcal{U}$. The max of $|f|$ over $\overline{\mathcal{U}}$ cannot occur in $\mathcal{U}$ by MMP, so it must occur on $\partial \mathcal{U}$. The case $f =$ constant is also clear.

Beware In general, $\min_{\overline{\mathcal{U}}} |f| \neq \min_{\partial \mathcal{U}} |f|$.