Math 220 A — Lecture 9

October 30, 2023

Theorem (Open Mapping Theorem)

$f: \mathcal{U} \longrightarrow \mathbb{C}$ not constant holomorphic $\Rightarrow$ f is open.

Proof Suffices to show $f(\mathcal{U})$ open. Else, if $V \subseteq \mathcal{U}$ open, work with $f/_V : V \longrightarrow \mathbb{C}$. This is not constant because of the identity principle.

else $f^{new} = f - f(a)$

Let $a \in \mathcal{U}$. We may assume $f(a) = 0$.

Claim $\exists\ r > 0$ such that $\Delta(0, r) \subseteq f(\mathcal{U})$. This would show $f(\mathcal{U})$ contains a neighborhood of $f(a) = 0$. $\Rightarrow$ $f(\mathcal{U})$ open.

disc centered at a

Proof Since $\mathcal{U}$ open $\Rightarrow$ $\exists\ \bar{\Delta}(a) \subseteq \mathcal{U}$. We may assume $f/_{\partial \bar{\Delta}(a)}$ has no zeros. (Argue by contradiction. This would give a sequence of zeros of f accumulating at a, a contradiction).

Let $r = \frac{1}{2} \min_{z \in \partial\overline{\Delta}(a)} |f(z)| > 0$.

Let $w \in \Delta(0,r)$. We need to show $\exists\, z \in U$, $f(z) = w$.

$\iff \exists$ zero z for $f - w$ in U. Assume otherwise and

let $g : U \longrightarrow \mathbb{C}$, $g = \frac{1}{f - w}$. This is holomorphic.

By Cauchy's estimate with $k = 0$ we find

$$|g(a)| \leq \max_{\partial\Delta(a,r)} |g| \iff$$

$$\iff \underbrace{|f(a)}_{0} - w| \geq \min_{\partial\Delta(a,r)} |f - w| \Rightarrow$$

$f(a) = 0$

However for $z \in \partial\Delta(a,r)$, we have

$w \in \Delta(0,r)$

$$|f(z) - w| \geq |f(z)| - |w| \underbrace{\geq}_{\text{def. of } r} 2r - |w| > |w|$$

$$\Rightarrow \min_{\partial\Delta(a,r)} |f(z) - w| > |w| = |\underbrace{f(a)}_{0} - w|, \text{ a contradiction}$$

This completes the argument.

Example $f: U \longrightarrow \mathbb{C}$, $P \in \mathbb{R}[X, Y]$ not constant

$$P(\operatorname{Re} f, \operatorname{Im} f) = 0 \implies f \text{ constant.}$$

Thus, taking $P = aX + bY - c$, we see

$$a \operatorname{Re} f + b \operatorname{Im} f = c \implies f \text{ constant.}$$

Proof By OMT, $f(U)$ is open so it contains a disc Δ.

Since $P(\operatorname{Re} f, \operatorname{Im} f) = 0 \implies f(U) \subseteq \{(x,y) : P(x,y) = 0\}$.

$\implies \Delta \subseteq \{(x,y) : P(x,y) = 0\}$ This cannot happen.

Indeed, write $P(x,y) = \sum_{k=0}^{N} a_k(x) y^k$, $a_N \not\equiv 0$.

Fix x such that $a_N(x) \neq 0$. (finitely many roots). For such x, y takes on at most N values for which $P(x,y) = 0$.

But if $\Delta \subseteq \{(x,y) : P(x,y) = 0\}$, for each x there would be ∞-many y's. contradiction.

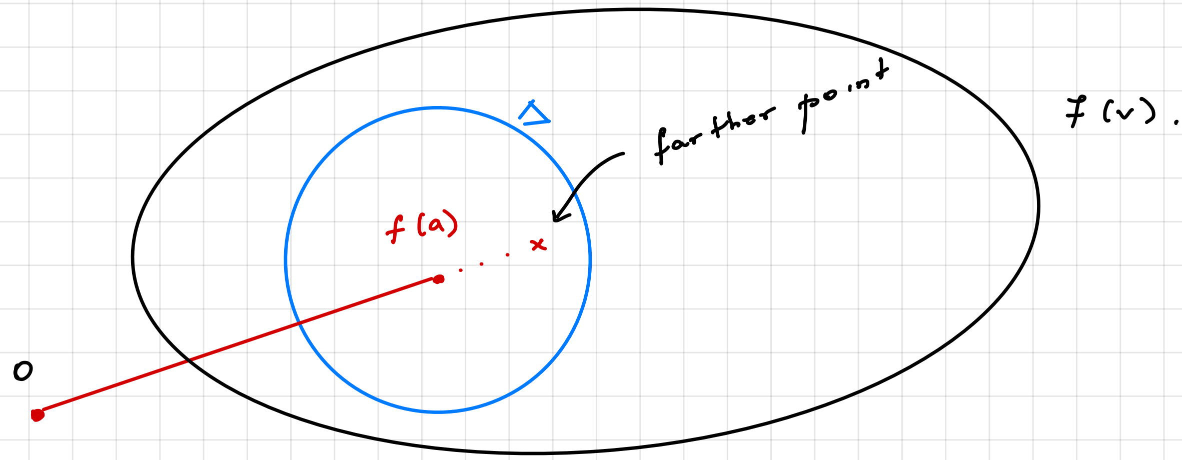
Theorem (Maximum Modulus) ↙ Conway IV. 3. 11

$f : \mathcal{U} \longrightarrow \mathbb{C}$ holomorphic, non constant. then $|f|$ cannot have a local maximum in $\mathcal{U}$.

Proof Assume that $|f|$ achieves a local maximum at a.

$\Rightarrow \exists\, V \ni a,\ V \subseteq \mathcal{U}$, $|f|\big/_V$ has a maximum at a.

By OMT, $f(V)$ is open. $\Rightarrow \exists$ disc Δ centered at $f(a)$ $\Delta \subseteq f(V)$. Note that $|f|$ measures distance from the origin. The disc Δ has points farther from 0 than $f(a)$ contradicting the assumption $|f|\big/_V$ has maximum at a.



2. Laurent Series (Conway V.1)

We have seen $f : \Delta(a,r) \longrightarrow \mathbb{C}$ then

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

— Taylor series

We consider Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

Convergence of Laurent series

$$f^+(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

$$f^-(z) = \sum_{k=-\infty}^{-1} a_k (z-a)^k = \sum_{k=1}^{\infty} a_{-k} (z-a)^{-k}$$

$$f(z) = f^+(z) + f^-(z).$$

Def f converges absolutely & uniformly provided f^+, f^- do so.

Remark

radius of convergence

f^+ converges if $|z-a| < R$.

f^- converges if $|z-a|^{-1} < r^{-1} \iff |z-a| > r$.

For power series, convergence is absolute & uniform on compact subsets.

Define $\Delta(a; r, R) = \{z : r < |z-a| < R\}$, $0 \le r < R \le \infty$.

Theorem Let $f : \Delta(a; r, R) \longrightarrow \mathbb{C}$ holomorphic. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$ can be expanded into

Laurent series, converging absolutely & uniformly on compact sets in $\Delta(a; r, R)$. Furthermore,

$$a_k = \frac{1}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{k+1}} \, dw. \quad \forall\, r < \rho < R.$$

Remark An important case is $r=0$. Then

$$\Delta^*(a, R) = \Delta(a, R) \setminus \{a\} = \text{punctured disc.}$$

$$f : \Delta^*(a, R) \longrightarrow \mathbb{C} \text{ holomorphic} \Rightarrow f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

Compare this to Taylor expansion.

Pierre Alphonse Laurent

1813 – 1854

(engineer in the army).

The original work on Laurent series was not published.

Cauchy writes (C.R. Acad. Sci. Paris, 1843, page 938)

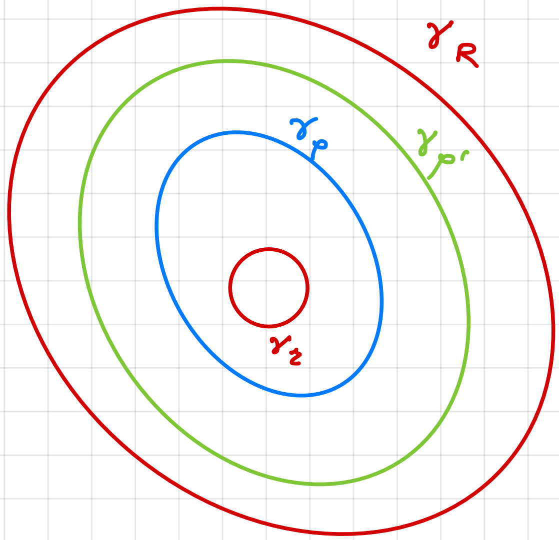
L'extension donnée par M. Laurent nous parait digne de remarque

(After Remmert, Complex Analysis, page 350)

Proof (of Laurent expansion) $A = \Delta(a; r, R)$.

$\boxed{a}$ WLOG $a = 0$; else work with $f(z+a)$.

$\boxed{b}$ the expression $a_k = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(w)}{w^{k+1}}\,dw$



is independent of ρ. Indeed

$\gamma_\rho \overset{A}{\sim} \gamma_{\rho'}$ and use

Cauchy Homotopy Theorem.

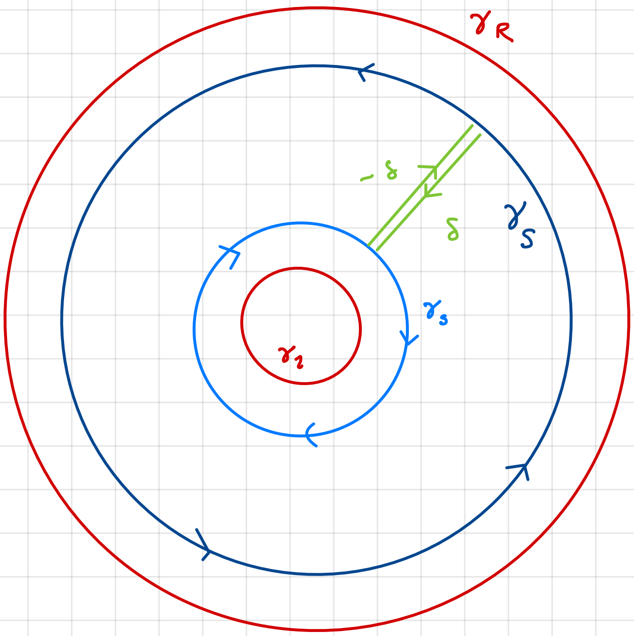
$\boxed{c}$ suffices to prove pointwise convergence in z.

Indeed, convergence of f $\Longleftrightarrow$ convergence of f^+ & f^- in $r < |z| < R$.

But then f^+ converges in $|z| < R$ (power series have discs of converge) & we remarked convergence is absolute & uniform on compacts. Same for f^-.

Pointwise convergence

Let $r < s < |z| < S < R$



Let δ be a segment joining γ_s, γ_S avoiding z.

Let

$$\gamma = \gamma_S + \delta + \gamma_s + (-\delta)$$

Note $\gamma \sim 0$. This can be seen by continuously shrinking $\delta \to 0$.

Also $n(\gamma, z) = 1$ since $n(\gamma_s, z) = 0$ as z is outside and $n(\gamma_S, z) = 1$ as z is interior to γ_S. $\Rightarrow n(\gamma, z) = 1$.

CIF:

$$(+) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z}\, dw.$$

$$= \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w - z}\, dw$$

The two terms will give the positive / negative parts of Laurent series.

Key expansions (Remember them) $r < |z| < S$.

$\boxed{i}$ over γ_S : $\dfrac{1}{w - z} = \dfrac{1}{w} \cdot \dfrac{1}{1 - \frac{z}{w}} = \displaystyle\sum_{k=0}^{\infty} \frac{1}{w} \left(\frac{z}{w}\right)^k$

$$= \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}. \quad (1)$$

The convergence is uniform in w since $\left|\frac{z}{w}\right| = \frac{|z|}{S} < 1$. We can define $M_k = \dfrac{|z|^k}{S^{k+1}}$, $f_k(z) = \dfrac{z^k}{w^{k+1}}$ and invoke Weierstrass M-test to conclude uniform convergence.

We can multiply by $f(w)$. Uniform convergence still holds. (Use $M_k = \dfrac{|z|^k}{S^{k+1}} \cdot \sup\limits_{\gamma_S} |f|$.)

We can then integrate term by term. (Rudin). Thus

$$\frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w - z} \, dw \overset{(1)}{=} \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_S} \frac{f(w)}{w^{k+1}} \, dw \cdot z^k$$

$$= \sum_{k=0}^{\infty} a_k z^k. \quad (*)$$

$\boxed{ii}$ over γ_s, we use a different expansion

$$\frac{1}{w - z} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{w}{z}} = \sum_{k=0}^{\infty} -\frac{1}{z}\left(\frac{w}{z}\right)^k$$

$$= \sum_{k=0}^{\infty} -\frac{w^k}{z^{k+1}}. \qquad (2)$$

Here $\left|\frac{w}{z}\right| = \frac{s}{|z|} < 1$. By the same arguments

$$-\frac{1}{2\pi i}\int_{\gamma_s} \frac{f(w)}{w - z}\, dw \overset{(2)}{=} \sum_{k=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_s} f(w)\, w^k\, dw \cdot z^{-k-1}$$

$$= \sum_{k=0}^{\infty} a_{-k-1}\, z^{-k-1}$$

$$= \sum_{k=-\infty}^{-1} a_k\, z^k. \qquad (**)$$

$(+)$, $(*)$, $(**)$. imply the Theorem.

$\boxed{3}$ Types of singularities

$$f : \Delta^*(a, R) \longrightarrow \mathbb{C}, \quad \text{holomorphic.}$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{Laurent series.}$$

Terminology

$\boxed{i}$ the coefficient of $(z-a)^{-1}$:

$$a_{-1} = \operatorname*{Res}_{z=a} f = \text{residue} = \frac{1}{2\pi i} \int_{\gamma_r} f \, dz$$

$\boxed{ii}$

$$\sum_{n=-\infty}^{-1} a_n (z-a)^n = \text{principal part.}$$

Three cases

$\boxed{A}$ $a_k = 0 \;\; \forall\, k < 0 \iff$ Taylor expansion

$\iff f$ extends holomorphically across a

Removable singularity

$\boxed{B}$ $a_k = 0 \;\; \forall\, k < -N, \quad a_{-N} \neq 0$

Pole of order N.

$\boxed{C}$ $a_k \neq 0, \; k < 0$ happens infinitely often

Essential singularity