Math 220A - Fall 2023 - Midterm

Name: ______________________________

Student ID: ___________________________

Instructions:

Please print your name and student ID.

You may use results **proved** in class without proof, unless the problem asks you to reprove such a result. Results which were **not proved** in class, e.g. Big Picard, are not allowed.

There are 5 questions which are worth 50 points. You have 80 minutes to complete the test.

Question	Score	Maximum
1		12
2		10
3		8
4		10
5		10
Total		50

Problem 1. [*12 points; 4, 4, 4.*]

Let

$$f(z) = \frac{1}{(z-3)(z-2)}.$$

(i) Write down the Laurent expansion of f in the region $2 < |z| < 3$.

(ii) Construct an explicit primitive of f in the region

$$U = \mathbb{C} \setminus \{x \in \mathbb{R} : 2 \leq x \leq 3\}.$$

Don't forget to justify why your answer is well-defined.

(iii) Let

$$U_1 = \{z : |z| < 2\}, \quad U_2 = \{z : 2 < |z| < 3\}, \quad U_3 = \{z : |z| > 3\}.$$

In which of these regions does f admit a primitive? Please justify your answer.

Problem 2. [*10 points; 4, 6.*]

(i) Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Assume that there exist constants $C, R > 0$ such that

$$|f(z)| \leq C|z|^d$$

for all $|z| \geq R$. Show that f is a polynomial.

(ii) Let p, q be polynomials, and let $f : \mathbb{C} \to \mathbb{C}$ be entire such that

$$f(z)^3 + p(z)f(z) + q(z) = 0.$$

Show that f is a polynomial.

Problem 3. [*8 points.*]

Consider $f : \Delta(0,1) \to \mathbb{C}$ holomorphic and nonconstant, and define $M(r) = \max_{|z|=r} \mathrm{Re} f(z)$ for $0 \leq r < 1$. Show that $M : [0,1) \to \mathbb{R}$ is strictly increasing.

Problem 4. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Assume that the function $g(z) = f(z) \cdot f\left(\frac{1}{z}\right)$ is bounded on $\mathbb{C} \setminus \{0\}$. Show that $f(z) = cz^m$ for some constant c and $m \geq 0$.

Problem 5. [*10 points.*]

Let $U \subset \mathbb{C}$ be an open connected set, and let $f, g : U \to \mathbb{C}$ be holomorphic functions with $|f(z)| = |g(z)|$ for all $z \in U$. Show that $f = ag$ for some $a \in \mathbb{C}$ with $|a| = 1$.