Math 220C, Problem Set 2. Due Friday, April 17.

1. (Subharmonic functions.) Show that if \( u_n : G \to \mathbb{R} \) are subharmonic/superhamonic converging locally uniformly to \( u : G \to \mathbb{R} \), then \( u \) is also subharmonic/superharmonic.

2. (Laplacian in polar coordinates.) Prove the following formula for the Laplacian in polar coordinates
\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]

3. (Subharmonic functions.) Assume that \( \phi : G \to \mathbb{R} \) is a function of class \( C^2 \) such that \( \Delta \phi \geq 0 \).

Let \( a \in G \) and \( \overline{\Delta}(a, R) \subset G \). Define for \( 0 \leq r < R \) the function
\[
h(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(a + re^{it}) \, dt
\]

(i) Show that \( h \) is non-decreasing.

Hint: Let \( u(r, t) = \phi(a + re^{it}) \). Show that
\[
\left(r \frac{\partial}{\partial r}\right)^2 h = \frac{1}{2\pi} \int_0^{2\pi} \left(r \frac{\partial}{\partial r}\right)^2 \phi \, dt.
\]

Using the expression for \( \Delta \) in polar coordinates, and the fact that the integral of \( \frac{\partial^2 u}{\partial \theta^2} \) vanishes, conclude that \( rh'(r) \) is non-decreasing. Conclude that \( h'(r) \geq 0 \).

(ii) Using (i), show that \( \phi \) is subharmonic
\[
\phi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(a + re^{it}) \, dt.
\]

(iii) If \( \phi \) is in fact harmonic in \( G \setminus \{a\} \), show that
\[
\left(r \frac{\partial}{\partial r}\right)^2 h = 0
\]

and conclude that
\[
h(r) = \alpha \log r + \beta.
\]

(iv) Which of the following functions are subharmonic? superharmonic? harmonic? neither?
(a) \( f(x, y) = x^2 + y^2 \)
(b) \( f(x, y) = x^2 - y^2 \)
(c) \( f(x, y) = x^2 + y \)

4. (Harmonic functions.)

(i) Assume \( f : G \to \mathbb{C} \) is nowhere zero and holomorphic. Show that \( \log |f| \) is harmonic in \( G \).
(ii) Show that
\[ \frac{1}{2\pi} \int_0^{2\pi} \log |re^{it} - \rho| \, dt = \max(\log r, \log |\rho|). \]

**Hint:** You may assume \( \rho = 1 \) (why?) The case \( r < 1 \) follows from part (i). The case \( r > 1 \) can also be reduced to the case \( r < 1 \) by simple manipulations. The case \( r = 1 \) requires the integral
\[ \int_0^\pi \log \sin x = -\pi \log 2. \]
You may assume this integral.

5. **(Subharmonic functions.)** For this problem, we extend the conventions given in class for subharmonic functions. A subharmonic function \( u : \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty\} \) is assumed upper semicontinuous, and required to satisfy the sub-mean value property.

Show that the series
\[ u(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} \log \left| z - \frac{1}{2^n} \right| \]
defines a subharmonic function \( u : \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty\} \).

6. **(Barrier functions.)** Let \( G \) be a region, \( \zeta_0 \in \partial G \), and let \( \ell \) be a half-line starting at \( \zeta_0 \) that intersects \( \overline{G} \) only at \( \zeta_0 \). Let \( \zeta_1 \neq \zeta_0 \) be a point on the half line \( \ell \).

Show that \( \zeta_0 \) is a barrier for \( \partial G \). Draw a picture and convince yourself this justifies the terminology “barrier.”

**Hint:** Consider
\[ \omega(z) = \text{Im} \ e^{i\alpha} \sqrt{\frac{z - \zeta_0}{z - \zeta_1}}, \]
for suitable \( \alpha \). Show that \( \omega \) is harmonic, \( \omega(\zeta_0) = 0 \) and all other boundary values are positive.

7. **(Dirichlet problem for the upper half plane.)** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous bounded function. Show that
\[ u(x + iy) = \frac{1}{\pi} \int_{\mathbb{R}} f(t) \frac{y}{(x - t)^2 + y^2} \, dt \]
defines a harmonic function in the upper half plane with boundary values equal to \( f \).

**Hint:** You may wish to use the Cayley transform.