Math 220C, Problem Set 3. Due Friday, April 24.

In this problem set, for a holomorphic function $f : G \to \mathbb{C}$ with $\Delta(0,r) \subset G$ we write

- $M(r) = \sup \{|f(z)| : |z| = r\}$
- $N(r)$ is the number of zeros of $f$ in $\Delta(0,r)$ counted with multiplicity.

1. (Jensen’s inequality.)
   (i) Let $f : G \to \mathbb{C}$ be holomorphic, and let $\Delta(0,r) \subset G$. Assume that $f(0) \neq 0$.
   Let the zeros of $f$ in the open disc $\Delta(0,r)$ be with multiplicities $z_1, \ldots, z_k$. Show that
   $\quad |f(0)| \leq |z_1 \ldots z_k| \cdot \frac{M(r)}{r^k}$.
   (ii) Assume $f : G \to \mathbb{C}$ is holomorphic, $\Delta(0,1) \subset G$ and $|f(z)| \leq 1$ for all $z \in G$.
   Assume
   $\quad f \left( \frac{1}{2} \right) = f \left( \frac{i}{2} \right) = 0$.
   Show $|f(0)| \leq \frac{1}{4}$.

2. (Jensen’s formula.) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.
   (i) If $f(0) = 1$ show that
   $\quad N(r) \log 2 \leq \log M(2r)$.
   (ii) Verify this by hand for the function $f(z) = \cos z$.
   (iii) Assume that
   $\quad |f(z)| \leq \exp(A|z|^k)$
   for $A > 0$ and $k$ natural number. Show that
   $\quad \limsup_{r \to \infty} \frac{\log N(r)}{\log r} \leq k$.

3. (Jensen’s formula.) Assume $f$ is a bounded holomorphic function
   $\quad f : \Delta(0,1) \to \mathbb{C}$
   with zeroes $a_1, a_2, \ldots$ listed with multiplicity. Using Jensen’s inequality, show that
   $\quad \prod_n |a_n| > 0$ and conclude
   $\quad \sum_n (1 - |a_n|) < \infty$.

4. (Order.) If $f, g$ are entire functions of orders $\lambda_1, \lambda_2$, show that $fg$ has order
   $\quad \leq \lambda = \max(\lambda_1, \lambda_2)$.

5. (Order and Genus.)
   (i) Show that $\cos z$ is a function of genus 1 and order 1.
   (ii) Show that $\cos \sqrt{z}$ is an entire function of order $\frac{1}{2}$ and genus 1.
6. (Exponent of convergence.) Let
\[ |a_1| \leq |a_2| \leq \ldots \]
be a sequence of non-zero complex numbers and let
\[ \alpha = \inf \left\{ t : \sum_n \frac{1}{|a_n|^t} < \infty \right\}. \]
Assume \( f \) is an entire function with zeros only at \( a_1, a_2, \ldots \). Let \( \lambda \) be the order of \( f \).

(i) Show that for any \( \epsilon > 0 \),
\[ \sum_n \frac{1}{|a_n|^\alpha + \epsilon} < \infty, \quad \sum_n \frac{1}{|a_n|^\alpha - \epsilon} = \infty. \]

(ii) Show that \( \alpha \leq \lambda \).

Hint: Fix \( \epsilon > 0 \) and show that \( \lambda + \epsilon > \alpha \). To this end, use Problem 2 to derive
\[ n \leq N(r = |a_n|) \leq \log M(2|a_n|)/\log 2. \]

On the other hand, use \( \log M(r) \leq r^{\lambda + \epsilon} \) for \( r \) sufficiently large. Find a bound on \( |a_n| \) and conclude.