Math 220C, Problem Set 4. Due Friday, May 1.

1. Let \( f, g \) be two entire functions of finite order \( \lambda \). Assume \( f(a_n) = g(a_n) \) for a sequence \( \{a_n\}_{n \geq 0} \) with
\[
\sum_{n=0}^{\infty} \frac{1}{|a_n|^{\lambda+1}} = \infty.
\]
Show that \( f = g \).

2. (i) Find all entire functions \( f \) of finite order such that \( f(\log n) = n \) for all integers \( n \geq 1 \).

(ii) Give an example of an entire function \( f \) with zeroes only at \( \log n \) for integers \( n \geq 1 \).

3. Let \( f(z) = \sum_{n=0}^{\infty} c_n z^n \) be an entire function of order \( \lambda \). Let
\[
\mu = \limsup_{n \to \infty} \frac{n \log n}{-\log |c_n|} > 0.
\]
Show that \( \lambda = \mu \).

(i) First show that \( \lambda \geq \mu \) by showing that for all \( \epsilon > 0 \) we have \( \lambda > \mu - \epsilon \).

Hint: By definition
\[
n \log n \geq -(\mu - \epsilon) \log |c_n|
\]
for infinitely many \( n \). Use Cauchy’s estimate for \( |c_n| \) to conclude that
\[
\log M(R) \geq n \log R - \frac{1}{\mu - \epsilon} n \log n
\]
for all \( R \). Use this for
\[
R = (en)^{\frac{1}{\mu - \epsilon}} \implies \log M(R) \geq \frac{n}{\mu - \epsilon} = \frac{R^{\mu - \epsilon}}{e(\mu - \epsilon)}.
\]
Conclude that \( \lambda \geq \mu - \epsilon \).

(ii) Conversely, show that \( \lambda \leq \mu \) by showing that \( \lambda < \mu + \epsilon \) for all \( \epsilon > 0 \).

Hint: If \( n \) is sufficiently large, \( |c_n| \leq n^{-\frac{n}{\mu+\epsilon}} \). Conclude that
\[
M(R) \leq \sum_{n} R^n n^{-\frac{n}{\mu+\epsilon}}.
\]
To estimate this series, break the sum into two pieces \( S_1, S_2 \) corresponding to
\( n < (2R)^{\mu+\epsilon} \) and \( n \geq (2R)^{\mu+\epsilon} \). Show
\[
S_2 = \sum_{n > (2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} < 1.
\]
Show
\[ S_1 = \sum_{n \leq (2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{\mu+\epsilon}} \leq R^{(2R)^{\mu+\epsilon}} \sum_{n \geq 1} n^{-\frac{n}{\mu+\epsilon}} \leq CR^{(2R)^{\mu+\epsilon}}. \]

Conclude.

4. Let \( a > 0 \). Show that the function
\[ f(z) = \sum_n z^n a_n \]
is entire and find its order.

5. If \( f \) is an entire function of finite order \( \lambda \), show that \( f' \) also has order \( \lambda \).

6. Let \( f(z) = \prod_{n=1}^{\infty} (1 - a^n z) \) for \( |a| < 1 \). Show that \( f \) has order 0.

*Hint*: Fix \( \epsilon > 0 \). Two estimates are needed:
\[ \log |1 - w| \leq C, \text{ for } |w| \geq 1/2 \]
and
\[ \log |1 - w| \leq C|w|^{\epsilon}, \text{ for } |w| \leq 1/2. \]