Plan for Math 220c

1. Harmonic functions ($\hat{X}$).
2. Order, genus of entire functions. Hadamard factorization ($\hat{X}$).
3. Little / big Picard ($\hat{X}^I$).
4. Extra topics: Riemann surfaces (if time).

Website: math.ucsd.edu/~dopera/220520.html

Grades: homework due on Fridays (5pm)

Qualifying Exam: TBA
Harmonic functions share many properties with holomorphic functions. (Integral formula, maximum module, convergence)

Recall \( G \subset \mathbb{C} \), \( u \in L^2(G) \), \( u : G \rightarrow \mathbb{R} \) is harmonic if

\[
\Delta u = u_{xx} + u_{yy} = 0
\]

Review 1) \([\mathbb{Z}20A]\) If \( f : G \rightarrow \mathbb{C} \) holomorphic \( \Rightarrow \text{Re} \, f \) and \( \text{Im} \, f \) are harmonic. They satisfy CR equations

\[
\begin{align*}
\bar{u}_x &= v_y \\
\bar{u}_y &= -v_x
\end{align*}
\]

2) \([\mathbb{Z}20B]\) Conversely, if \( G \) is simply connected, any harmonic function \( u \) arises this way. That is, if \( f : G \rightarrow \mathbb{C} \) holomorphic with \( u = \text{Re} \, f \).

Remark If \( u \) is harmonic, \( u : G \rightarrow \mathbb{R} \) \( \Rightarrow u \) is \( C^2 \). (Regularity)

Proof This is a local statement. Take any \( \bar{a} \in G \), \( \bar{D}(a,r) \subset G \). Suffices to show \( u \) is \( C^2 \) in \( \bar{D}(a,r) \). By \([\mathbb{Z}20B]\), \( u = \text{Re} \, f \), \( f \) holomorphic in \( \bar{D}(a,r) \). We have seen \( f \) is \( C^2 \) \( \Rightarrow u \in C^2 \).
Properties of harmonic functions

Today:
1. Mean Value Property (MVP)
2. Maximum Modulus Principle (MMP)

Next time:
3. Poisson integral formula
4. Dirichlet problem (for disc, maybe more?)
1. **Mean Value Property (MVP)**

**Theorem:** Let \( u : \mathbb{C} \to \mathbb{R} \) be harmonic, \( \Delta (a, r) \subseteq \mathbb{C} \). Then \( u \) satisfies

\[
u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta.
\]

**Proof:** Let \( p > r \) be chosen so \( \Delta (a, r) \subseteq \Delta (a, p) \subseteq \mathbb{C} \). In \( \Delta (a, p) \), write \( u = \text{Re} f \), \( f \) holomorphic in \( \Delta (a, p) \).

By Cauchy Integral Formula (CIF), we have

\[
f(z) = \frac{1}{2\pi i} \oint_{\partial \Delta (a, r)} \frac{f(w)}{w-z} \, dw.
\]

We have

\[
\frac{d2}{i(2-\alpha)} = dt.
\]

Then taking real part

\[
u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta, \text{ as needed.}
\]

**Def:** \( u : \overline{\mathbb{C}} \to \mathbb{R}, u \in L^2(\mathbb{C}) \), satisfies MVP provided \( u \) a.e., \( \Delta (a, r) \subseteq \mathbb{C} \), we have

\[
u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta.
\]

center average of boundary values
Maximum (modulus) principle

Theorem 2. \( u : \mathbb{R} \to \mathbb{R}, u \in \mathcal{C}^2(\mathbb{R}) \) satisfies \( \text{MVP} \). Assume \( \exists \ a \in G \) such that

\[ u(a) > u(x) + 2\epsilon G. \]

Then \( u \) is constant.

Remark. We have seen if \( u \) harmonic \( \implies u \text{ MVP} \). \( \iff u \text{ satisfies } \text{MVP}. \)

Proof. Without loss of generality, \( u(a) = 0 \) if \( u(a) \neq 0 \), then work instead with the function

\[ w_{\text{new}} = u - u(a). \]

\( w_{\text{new}} \) still satisfies \( \text{MVP} \).

Want \( w \equiv 0 \). Let

\[ S_L = \{ x \in G : u(x) = 0 \}. \]

\begin{enumerate}
\item \( S_L \neq \emptyset \) since \( a \in S_L \).
\item \( S_L = \) closed in \( G \), because \( u \) continuous \( \implies S_L \subseteq G \) compact.
\item \( S_L \) open in \( G \).
\end{enumerate}

Proof of (c). Let \( x_0 \in S_L \). \( \exists \ \overline{\Delta}(x_0, r) \subseteq G. \) We show \( \Delta(x_0, r) \subseteq S_L \).

Let \( x \in \Delta(x_0, r) \). Let \( r = 12 \alpha - \alpha^1 \). Thus

\[ u = x_0 + r e^{i\theta}. \]
We know $2 \times \text{sgn} = u(2) = 0$. By mean value

$$0 = u(2) = \frac{1}{2\pi} \int_{0}^{2\pi} u(2 + p e^{it}) \, dt \quad \Rightarrow \quad \int_{0}^{2\pi} f(t) \, dt = 0$$

Lemma $f : [0, 2\pi] \to \mathbb{R}$ continuous, $f(t) \leq 0$ & $\int_{0}^{2\pi} f(t) \, dt = 0$. Then $f \equiv 0$. [HWK, Rudin].

Apply this to $f(t) = u(2 + p e^{it}) \leq u(2) = 0$. By lemma $f(t) \equiv 0$ & $f(t) \equiv 0 \Rightarrow u(2 + p e^{it}) = 0 = u(2) = 0 = x \in \mathbb{S}$. \( \triangleright \)

Remark If $u$ has MVE $\Rightarrow \bar{u} = \text{constant}$. Minimum modulus property.

Thm $u : G \to \mathbb{R}$ continuous & MVE. If $u$ has a minimum in $G \Rightarrow u \text{ constant}$. 
Eine harmonische Function u kann nicht in einem Punkt im Innern ein Minimum oder ein Maximum haben, wenn sie nicht überall constant ist.

(A harmonic function u cannot have either a minimum or a maximum at an interior point unless it is constant.)

“Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grosse,” Dissertation (1851)
Assume $G \subseteq \mathbb{C}$ (bounded or unbounded), $u, v : G \to \mathbb{R}$ bounded continuous functions satisfying 

\[
\limsup_{z \to a} u(z) \leq \liminf_{z \to a} v(z). \quad \text{Then either}
\]

\begin{itemize}
  \item $u < v$ everywhere on $G$
  \item $u \equiv v$ everywhere on $G$
\end{itemize}

**Proof** Let $\varphi = u - v$. Then $\varphi$ satisfies MVP and furthermore

\[
\limsup_{z \to a} \varphi(z) = \limsup_{z \to a} (u - v)(z) \leq \limsup_{z \to a} u(z) + \limsup_{z \to a} (-v(z)) = \limsup_{z \to a} u(z) - \liminf_{z \to a} v(z) \leq 0.
\]

**Rephrase** \[
\limsup_{z \to a} \varphi(z) \leq 0 \quad \text{with } \varphi < 0 \text{ on } a \text{ or else } \varphi \equiv 0.
\]

**Suffices** to show $\varphi < 0$ in $a$. (Because Version I).

Assume that $a$ with $\varphi(a) > 0$. Let $\varepsilon = \varphi(a) > 0$.

Let $K = \{ z : \varphi(z) \geq \varepsilon \}$. Since $a \in K \Rightarrow K \neq \emptyset$.

**Claim** $K$ is compact.

Then let $z_0$ be the point where $\varphi$ achieves a maximum in $K$,

$\Rightarrow z_0$ is a maximum for $\varphi$ in $C$ since in $K$, $\varphi \geq \varepsilon$ and outside $K$, $\varphi < \varepsilon$. But the existence of a max. in $C$ is impossible by Version I.

**Proof of the claim** Suffices to show $K$ bounded & closed in $C$.

For $a \in \mathbb{C}$,

\[
\limsup_{z \to a} \varphi(z) \leq 0 \Rightarrow \exists \Delta(a, \varepsilon_a) \text{ such that } \varphi < \varepsilon \text{ in } \Delta(a, \varepsilon_a). \quad \text{By Lebesgue covering lemma for the covering}
\]
\[ \Delta (a, r_a) \] of \( \mathbb{R}^n \), we can find \( \delta > 0 \) s.t. all discs \( \Delta (a, \delta) \subseteq \Delta (a', \delta') \).

Thus \( K \subseteq \{ z : d(z, \mathbb{R}^n) \geq \delta \} \). Then \( K \) is compact.

(You can also prove this directly w/o Lebesgue).

**Important Corollary.** If \( G \) is bounded, \( f : \overline{G} \rightarrow \mathbb{R} \) continuous & MVE.

\[ f \equiv 0 \text{ on } \partial G \implies f \equiv 0 \text{ in } G. \]

**Proof.** This follows from MME Version \( \overline{G} \), applied twice. Let \( a \in \partial G \).

\[
\begin{cases}
  u = f \\
  v = 0
\end{cases}
\]

\[ 0 = \limsup_{x \to a} f(x) \leq \liminf_{x \to a} u(x) = 0 \implies v \equiv 0 \quad (\text{w} / \text{e} \ f \text{ cont. on } \partial G, f = 0 \text{ on } \partial G). \]

\[
\begin{cases}
  u = 0 \\
  v = f
\end{cases}
\]

\[ 0 \leq \liminf_{x \to a} f(x) = 0 \implies v \equiv f \geq 0. \]