Math 220C - Lecture 11

May 6, 2020
Qualifying Exam - May 27 - 1-4 PM.

Logistics - Friday

Topics - see website

Last time - Little Picard

Step A (Landau):

\[ f: \mathbb{C} \to \mathbb{C} \text{ avoids } 0, 1, \mathbb{C} \text{ simply connected then.} \]

\[ f = \frac{1}{2} (1 + \cos \pi \cos \pi g). \text{ Take } \mathbb{C} = \mathbb{C}. \]

- \( \text{Im } g \) contains no disc of radius 2.
- \( \frac{g(R)}{g'(w)} \) contradicts Bloch's theorem.

Step B (Bloch):

\( f: \text{holomorphic in } \overline{\mathbb{C}}, \ f'(z) = 1 \Rightarrow \text{Im } f \text{ contains a disc of radius } L \neq 0. \]

\[ L = \frac{3}{2} \sqrt{2} - 2 \text{ works (last time).} \]
Today: Great Picard.

- Schottky's theorem (§11.3).
- Montel's second theorem (§11.4).
- Conclusion of the proof.

All these concern the family \( F = \{ f : G \to \mathbb{C} \text{ omits } 0 \} \).

### Schottky's Theorem

**Thm** There is a universal function \( C(\alpha, \beta) \), increasing in \( 0 < \alpha < \infty, 0 < \beta < 1 \), such that

\[ |f(z)| \leq C |f(z_0)|, \quad |z| \leq 121 \leq |z_0| \]

**Remark** If \( |f(z_0)| = \infty \), then \( |f(z)| \leq C(\alpha, \beta) \) for \( 121 \leq |z| \).

This theorem controls the growth of functions in the family \( F \).

**Remark** A possible function is:

\[ C(\alpha, \beta) = \exp(\pi \exp \pi \left(3 + 2\alpha + \frac{2\beta}{1-\beta}\right)) \]

We will deduce this from Bloch's theorem.

### Claim

Given \( z \in \mathbb{C} \), \( \cos \pi z = 2 \). We can find \( a \) such that

\[ |a| \leq |a + 12| \]

**Proof of claim** \( a \to a + 2 \). WLOG we may assume \( \Re a \in [-1,1] \).

We show \( |\Im a| \leq |\cos \pi a| = 121 \). Thus \( |a| \leq |\Re a| + |\Im a| = 1 + 121 \).

\[ a = x + iy \quad \Rightarrow \quad |\cos \pi (x + iy)|^2 = \cos^2 \pi x + \sinh^2 \pi y \geq \sinh^2 \pi y \geq (\pi y)^2 \geq |\Im a|^2 = |\Im a|. \]
Proof. Take any \( f \in \mathcal{O}(\bar{D}) \). In some neighborhood of \( \bar{D} \), \( f = \frac{1}{2} (1 + \cos \pi F) \).

\[
\Rightarrow \cos \pi F(z) = 2 f(z) - 1.
\]
By claim we can modify \( F(z) \) so that

\[
|F(z)| \leq 1 \Rightarrow 2 f(z) - 1.
\]

\[
\Rightarrow \cos \pi F(z) = 2 f(z) - 1.
\]
Write \( F = \cos \pi g \). By claim we can arrange

\[
|g(z)| \leq 1 + |F(z)| \leq 1 + |2 f(z) - 1| \leq 3 + 2 |f(z)| = 3 + 2 \alpha.
\]

Assume \( |z| \leq \rho \). \( \Rightarrow \bar{D}(z, 1 - \rho) \subseteq \bar{D} \). Define

\[
\begin{align*}
\hat{h}(w) &= g(z + (1 - \rho) w) \quad \text{in the unit disc} \ \bar{D}.
\int_{\rho} g'(z) \mathrm{d}w.
\end{align*}
\]

\[
h'(z) = 1. \Rightarrow \text{Im} h \text{ contains a disc of radius } 1.
\]

\[
\Rightarrow \text{Im} g \text{ contains a disc of radius } (1 - \rho) |g'(z)|.
\]

\[
\text{Im} g \text{ cannot contain a disc of radius } 2. \text{ (last time) } \Rightarrow
\]

\[
|1 - \rho| g'(z)| \leq 2. \Rightarrow |g'(z)| \leq \frac{2}{|1 - \rho|}.
\]

\[
1 g(z) - g(\omega) = \int_{\omega}^{2} g'(z) \mathrm{d}w \leq \int_{0}^{2} \frac{2}{|1 - \rho|} \mathrm{d}w = \frac{2}{L(1 - \rho)} \rho.
\]

\[
\Rightarrow |g(z)| \leq 3 + 2 \alpha + \frac{4 \rho}{L(1 - \rho)}. \quad (*)
\]

\[
\Rightarrow |f(z)| = \frac{1}{2} |1 + \cos \pi \cos \pi g(z)| \leq \exp (|1| \cos \pi g(z)) = \exp (|1| \exp \pi |g(z)|).
\]

\[
(1) \leq \exp (\pi \exp \pi |g(z)|).
\]

\[
(*) \leq \exp (\pi \exp \pi (3 + 2 \alpha + \frac{4 \rho}{L(1 - \rho)})).
\]

Remark: \( \frac{1}{2} |1 + \cos \omega| \leq e^{10 |\omega|} \). \( (1) \)

\[
|\cos \omega| \leq e^{20 |\omega|}. \quad (2)
\]

\( (2) \Rightarrow (1) \) double angle formula

\[
(2): |\cos \omega| = \frac{1}{2} |e^{i \omega} + e^{-i \omega}| \leq \frac{1}{2} (|e^{i \omega}| + |e^{-i \omega}|) = \frac{1}{2} (e^{i \omega} + e^{-i \omega}) \leq e^{10 |\omega|} \cdot \omega_1.
\]
Friedrich Schottky
(1851 - 1935)

Academic advisors
Karl Weierstrass
Hermann von Helmholtz

He worked on elliptic, abelian, and theta functions.

Schottky problem:
Characterization of Jacobian varieties amongst abelian varieties.

The author is of a clumsy appearance, unprepossessing, a dreamer, but if I’m not completely wrong, he possesses an important mathematical talent. [...] As rector I had to cancel his name from the register because neither had he attended lectures nor were his whereabouts in Berlin known. (Weierstrass.)

\[
\text{Given } \alpha > 0 \text{ and } 0 < \beta < 1, \exists C(\alpha, \beta), \text{ such that for all } f : \Delta \to \mathbb{C} \setminus \{0, 1\} \text{ } \quad \frac{|f(0)|}{\alpha} \leq f(z) \leq C(\alpha, \beta) \text{ for } |z| \leq \beta.
\]
**Main Result**

Let \( G \subseteq C \), \( \mathcal{F} = \{ f : G \to C \setminus \{0, 1\} \} \) is normal in the sense that all sequences in \( \mathcal{F} \) admit subsequences converging locally uniformly to some function or to \( \infty \).

**Recall**

Usual Montel theorem: a family \( \mathcal{F} \) is locally bounded \( \iff \) \( \mathcal{F} \) normal.

**Proof.**

Let \( 2 \circ \in G \). Define

\[
\mathcal{F}^+ = \{ f \in \mathcal{F}, \quad |f(2)\circ| \leq 1 \},
\]
\[
\mathcal{F}^- = \{ f \in \mathcal{F}, \quad |f(2)\circ| \geq 1 \}.
\]

**Claim.** \( \mathcal{F}^+ \) is locally bounded.

Pick \( \{f_n\} \) a sequence in \( \mathcal{F} \). Then

\[ a) \ \text{many} \ f_n's \ in \ \mathcal{F}^+. \text{Since} \ \mathcal{F}^+ \ is \ locally \ bounded, \]

by usual Montel, \( \exists \) subsequence \( f_n \xrightarrow{\mathcal{F}} f \). Done.

\[ b) \ \text{many} \ f_n's \ in \ \mathcal{F}^- \ then \ 1/f_n \in \mathcal{F}^+ \]

Therefore \( 1/f_n \xrightarrow{\mathcal{F}} g \). But \( 1/f_n \) doesn't vanish. By Hurwitz's convergence \( \lim_{n \to \infty} f_n = g \) doesn't vanish or \( g \equiv 0 \).

- If \( g \) doesn't vanish \( 1/f_n \xrightarrow{\mathcal{F}} g \Rightarrow f_n \xrightarrow{\mathcal{H}} g \).
- If \( g \equiv 0 \Rightarrow 1/f_n \xrightarrow{\mathcal{F}} 0 \Rightarrow f_n \xrightarrow{\mathcal{H}} \infty. \)

Students:

Henri Cartan
Jean Dieudonné

Montel introduced and developed the notion of normal family.

He published the theorem named after him in his thesis in 1907. In 1927 he published a monograph on normal families.

Une suite infinie de fonctions analytiques et bornées a l'intérieur d'un domaine simplement connexe, admet au moins une fonction limite a l'intérieur de ce domaine.

(An infinite sequence of functions that are analytic and bounded in the interior of a simply connected domain admits at least one limit function in the interior of this domain.)

P. Montel, 1907

The family \( \tilde{f} = \{ f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\} \} \) is normal in \( G \).
Proof of claim: \( \mathcal{F} \) is locally bounded. We have bounds at \( z_1 \) already.

Let \( a \in G \). Let \( \gamma \) be a path in \( G \) from \( a \) to \( z_0 \). Find

\[ z_0, z_2, \ldots, z_n \text{ points on } \gamma. \]

Find discs \( \Delta_0, \Delta_1, \ldots, \Delta_n \) covering \( \text{Im } \gamma \).

\( \overline{\Delta_0}, \ldots, \overline{\Delta_n} \subseteq \mathbb{C} \) and \( z \in \overline{\Delta_0}, 2 \in \overline{\Delta_2} \).

Let \( R_0, \ldots, R_n \) be the radii.

Apply Schottky's theorem in \( \overline{\Delta_0}(z_0, R_0) \).

\[ |f(z_1)| \leq C \cdot \left( \left| f(z_0) \right|, \frac{1}{R_0} \right)^{12z_2 - 2z_1} \leq C_n \text{ (independent of } f \). \]

Schottky in \( \overline{\Delta}(z_2, R_2) \).

\[ |f(z_2)| \leq C \cdot \left( \left| f(z_1) \right|, \frac{1}{R_2} \right)^{12z_3 - 2z_2} \leq C_2 \text{ (independent of } f \right). \]

Repeat ... \[ |f(z_n)| \leq C_n. \] \[ \Rightarrow \] \[ |f(z)| \leq C_{n+1} \text{ for all } z \in \overline{\Delta}(z, \frac{1}{2} R) \]

This shows uniform boundedness near \( a \), so everywhere. \( \square \).

Great Picard

Thm: \( f: G \backslash \{a\} \to \mathbb{C} \) essential singularity at \( a \), then for all \( \Delta^*(a, R) \subseteq G \), \( f|\Delta^*(a, R) \) takes on all values, with at most one exception, \( \alpha \) - many times.

Proof: It suffices to show \( f|\Delta^*(a, R) \) omits at most one value.

WLOG \( f|\Delta^* \) omits 0 and 1. Thus look at \( \mathcal{F} = \{ f: \Delta^* \to \mathbb{C} \backslash \{0,1\} \}. \) This is normal.

by Montel. \( \square \).
Let $f_n(t) = f\left(\frac{2}{n}\right)$, sequence in $\mathcal{F}$. Thus $\exists$ subsequence $f_{n_k} \xrightarrow{l.u.} q$ or $f_{n_k} \xrightarrow{l.u.} \infty$.

Use this over compact set $l_2 = R/2$.

We deal with the first case. The second case work with $1/f_{n_k}$.

Since $f_{n_k} \xrightarrow{u} q$ in $l_2 = R/2$. \Rightarrow \left| f_{n_k}(x) - q(x) \right| + \left| q(x) \right| \leq 2M.

for some $M > 0$ and $k \gg 0$, for all $l_2 = R/2$.

\Rightarrow \left| f\left(\frac{2}{n_k}\right) \right| \leq 2M$ if $l_2 = R/2$.

\Rightarrow \left| f(x) \right| \leq 2M$ if $R/2 \leq l_2 \leq R/2n_k$ for $k \gg 0$.

By MMP

$\left| f(x) \right| \leq 2M$ if $R/2n_k \leq l_2 \leq R/2n_k$.

\Rightarrow $\left| f(x) \right| \leq 2M$ if $\delta \leq \epsilon \leq R/C$.

\Rightarrow $f$ has removable singularity, Contradiction. \Box

**Next:**

1) **Riemann Surfaces.** (sheaves, complex mfds), Chp I?

2) **Elliptic functions.** (Ahlfors)?

3) **Prime Number theorem (Zang)?**

E-mail me your preference.