**Last time** \( f: \mathbb{C} \rightarrow \mathbb{C} \) entire function

**Question** Relationship between
\[
\{ \text{growth of } f \} \leftrightarrow \{ \text{distribution of zeros of } f \}
\]

**Define** \( M(R) = \sup_{|z|=R} |f(z)| \)

- \( N(R) = \# \text{ zeros of } f \text{ in } \Delta(0, R), \) (with multiplicity).

**Define** The order of \( f \)
\[
\lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}
\]

**Recall**
1. \( \lambda(fg) \leq \max(\lambda(f), \lambda(g)) \) \((\text{HWK})\)
2. \( \lambda(z^m) = 0 \) \((\text{check})\).
3. \( \lambda(z^n) = d \) \((\text{check})\) \(f\) polynomial of degree \( d \)

**Remarks** In concrete terms, \( \forall \varepsilon > 0 \)

1. \( \exists R_\varepsilon > 0 \) s.t. \( |f(z)| < \varepsilon^{12} \) whenever \( |z| > R_\varepsilon \).
2. \( \forall R \exists |z| > R \) s.t. \( |f(z)| > \varepsilon^{12} \lambda - 2 \) \((\text{by definition})\)
3. \( \# \text{ zeros of } f \) \( f: \mathbb{C} \rightarrow \mathbb{C} \) entire, \( f(0) \neq 0 \), \( \# \text{ zeros of } f \) are
\[
|a_1| \leq |a_2| \leq \ldots \leq |a_n| \leq \ldots \quad \text{counted with multiplicity.}
\]
$N(R) = \# \{ j : |a_j| < R \}$.

We

$\alpha = \text{exponent of convergence/critical exponent}.$

$\alpha = \inf \{ t > 0 : \sum_{j=1}^{\infty} \frac{1}{|a_j|^t} < \infty \}$.

**Remark**

Last time, we saw if $f(0) = 1,$

$$N(R) < \log M(3R), \quad N(R) \log 2 < \log M(2R).$$

Want: $M \leq \alpha.$

Can show: $N \leq \alpha.$ In fact $\alpha = \limsup_{R \to \infty} \frac{\log N(R)}{\log R}.$

**Example**

$a_n = n^2, \quad N(R) = \{ j : |a_j| < R \} \sim R^{1/3}.$

$$\alpha = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{|a_n|^t} = \sum_{n=1}^{\infty} \frac{1}{n^{3t}} < \infty, \quad 3t > 1, \quad t > \frac{1}{3}.$$

Confirms.$$

**Recall**

Weierstrass factorization theorem (2208).

$f : \mathbb{C} \to \mathbb{C}$ entire with zeros at $a_1, a_2, \ldots, a_n, \ldots, a_j \neq 0, \ldots$

$$f(z) = \prod_{n=1}^{\infty} E \left( \frac{z}{a_n} \right)$$

locally absolutely uniformly.

$$E_p(z) = \begin{cases} \frac{1}{z} & \text{if } p = 0 \\ (1-z)^p \exp \left( \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^p}{p} \right) & \text{if } p > 0. \end{cases}$$

where $\sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n + 1} < \infty.$ This is always possible $p_n = n.$

We need further restrictions.
Karl Weierstrass (1815 – 1897) was a German mathematician often cited as the "father of modern analysis".

Despite leaving university without a degree, he studied mathematics and trained as a teacher, eventually teaching mathematics, physics, botany and gymnastics.

Factorization Theorem (1876) $f: \mathbb{C} \rightarrow \mathbb{C}$ entire can be represented as

$$f(z) = z^m e^{\sum_{n=1}^{\infty} \frac{a_n}{z^n}}$$

where the elementary factors are

$$E_p(z) = \begin{cases} 1 - \frac{z}{p} & \text{if } p = 0 \\ (1 - z) e^{\frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}} & \text{if } p > 0 \end{cases}$$
Assume \( g \) is a polynomial of growth \( \sum \frac{1}{\ln n^p} < \infty \). The smallest \( p \) is called rank.

\[ \Rightarrow p+1 \geq \alpha \geq p. \]

The smallest \( \alpha \) is called rank.

Example \( g = \max (p, q) \). \( t = \infty \).

\[ f(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^n}. \]

\[ p = 0 \text{ works in } \# \), \( \sum_{n=1}^{\infty} \frac{Z^n}{2^n} = \sum_{n=1}^{\infty} 2^n < \infty \]

\[ p = 0, \ z = -\infty, \ \eta = 0 \]

\[ \sin z = z \sum_{n \neq 0}^{\infty} \frac{z^n}{n \pi} \]

\[ = 2 \sum_{n \neq 0}^{\infty} \frac{z^n}{n \pi} \exp (\frac{Z}{n \pi}) \exp (-\frac{z}{n \pi}) \]

\[ = 2 \sum_{n \neq 0}^{\infty} \exp (\frac{Z}{n \pi}) \Rightarrow p = 1 \Rightarrow t = 2. \]

\[ p = 1 \text{ works } \sum_{n \neq 0}^{\infty} \frac{1}{(n \pi) e < \infty \text{ for } (**) \}

Question A Knowing order \( \alpha \) \( \Rightarrow \) what is the genus \( f \)?

Thm (Hadamard) \( 0 \leq \alpha \leq \eta + 1 \). \( \Rightarrow \eta \) is uniquely determined.

Question B Knowing the genus \( \Rightarrow \) what can we say about the order?

Fact: \( \alpha = \lim_{n \to \infty} \exp \left( \frac{Z}{a_n} \right) \) = critical exponent. (We will not use.)
Jacques Hadamard
1865 - 1963 (age 97)

Proved the prime number theorem

Institutions

University of Bordeaux
Sorbonne
College de France
École Polytechnique
École Centrale Paris

Doctoral advisor
Émile Picard

Doctoral students
Maurice Fréchet
Marc Krasner
Paul Lévy
André Weil

\[ f(z) = \sum_{n=0}^{\infty} c_n z^n \]

where \( g \) polynomial of degree \( \leq \gamma \), \( p = \lceil \frac{\gamma}{2} \rceil \).

\[ f \text{ entire functions of order } \gamma, \text{ genus } h \]

\[ h \leq \gamma \leq h + 1 \]
Lemma. \( \log |E_p(w)| \leq C_p w^{p+2} \) for some constant \( C_p > 0 \).

Proof. \( p=0: \log 1-w \leq \log (1+w) \leq 1w. \)

We induct on \( p \). \( p-1 \rightarrow p \). We have two cases.

1) If \( |w| \geq \frac{1}{2} \):

\[
\log |E_p(w)| = \log |E_{p-1}(w)| + \log \left| \exp \left( \frac{w^p}{p} \right) \right|
\]

Induction

\[
\leq C_{p-1} |w|^{p-1} + \log \exp \Re \left( \frac{w^p}{p} \right).
\]

\[
= C_{p-1} |w|^{p-1} + \Re \left( \frac{w^p}{p} \right).
\]

\[
\leq C_{p-1} |w|^{p-1} + \frac{1}{p} |w|^p = \left( C_{p-1} + \frac{1}{p} \right) |w|^{p+\frac{1}{2}}
\]

\[
\leq 2 \left( C_{p-1} + \frac{1}{p} \right) \frac{|w|^{p+1}}{C_p}.
\]

Take \( C_p > 2 \left( C_{p-1} + \frac{1}{p} \right) \).

2) If \( |w| \leq \frac{1}{2} \):

\[
E_p(w) = (1-w) \exp \left( w + \frac{w^2}{2} + \ldots + \frac{w^p}{p} \right)
\]

uses

\[
\log (1-w) = \sum_{r=2}^{\infty} \frac{w^r}{r} = -w + \frac{w^2}{2} + \frac{w^3}{3} + \ldots
\]

\[
\log |E_p(w)| = \Re \left( \frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \ldots \right)
\]

\[
\leq \left| \frac{w^{p+1}}{p+1} \right| + \left| \frac{w^{p+2}}{p+2} \right| + \ldots
\]

\[
\leq |w|^{p+1} + |w|^{p+2} + \ldots
\]

\[
= |w|^{p+1} \left( 1 + |w| + \ldots \right) \leq 2 |w|^p.
\]

\[
\leq 2 |w|^{p+1} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots \right) = 2 |w|^{p+1}.
\]

Take \( C_p = \max(2, 2(C_{p-1} + \frac{1}{p})) \).

Take \( C_p > 2 \).
Proof that \( \lambda \leq h+1 \)

We know \( f = 2^m \) s.t.

\[
\prod_{n \geq 1} E_p \left( \frac{x}{a_n} \right)
\]

Suffices to show \( \prod_{n \geq 1} E_p \left( \frac{x}{a_n} \right) \) has order \( p+1 \).

We estimate

\[
\log \left| \prod_{n \geq 1} E_p \left( \frac{x}{a_n} \right) \right| = \sum_{n \geq 1} \log \left| E_p \left( \frac{x}{a_n} \right) \right| \leq
\]

Lemma

\[
\leq \sum_{n \geq 1} c_p \left| \frac{x}{a_n} \right|^{p+1} = C \cdot 2^{p+1}
\]

where \( C = c_p \sum_{n \geq 1} \frac{1}{|a_n|^{p+1}} < \infty \).

Thus

\[
\log M(R) \leq CR^{p+1}, \quad \text{where } M(x) \text{ is computed for } (\star)
\]

\[
\log \log M(R) \leq \log C + (p+1) \log R.
\]

\[
\Rightarrow \lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R} \leq \lim_{R \to \infty} \frac{\log C + (p+1) \log R}{\log R} = p+1.
\]

\[ \square \]

The inequality \( \lambda \geq h \) will be proved next time.