Math 220C - Lecture 7

April 22, 202
1. Last time: \( f: \mathbb{C} \rightarrow \mathbb{C} \) entire function

**Question** Relationship between

\[ \{ \text{growth of } f \} \leftrightarrow \{ \text{distribution of zeros of } f \} \]

**Define** \( M(R) = \sup_{|z|=R} |f(z)| \)

- \( N(R) = \# \text{ zeros of } f \in \Delta(0, R) \) (with multiplicity).

**Define** The order of \( f \)

\[ \lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{-\log R}. \]

**Recall**
- \( \lambda(fg) \leq \max(\lambda(f), \lambda(g)) \) (H*W*K)
- \( \lambda(z^n) = 0 \) (check).
- \( \lambda(z^d) = d \) (check) \( z \) polynomial of degree \( d \)

**Remarks** In concrete terms, \( \lambda \geq 0 \).

- \( \exists R_\varepsilon > 0 \text{ s.t. } |f(z)| < 2^{-12^{\lambda-2}} \) whenever \( |z| > R_\varepsilon \).
- \( \forall R \exists |z| > R \text{ s.t. } |f(z)| > 2^{-12^{\lambda-2}} \) (by definition)
- \( \exists \text{ zeros of } f \) \( f: \mathbb{C} \rightarrow \mathbb{C} \) entire, \( f(0) \neq 0 \), zeros of \( f \) are \( |a_1| \leq |a_2| \leq \ldots \leq |a_n| \leq \ldots \) counted with multiplicity.
\[ N(R) = \# \left\{ j : |a_j| < R \right\}. \]

\[ \alpha = \text{exponent of convergence/critical exponent}. \]

\[ \alpha = \inf \left\{ t > 0 : \sum_{j=1}^{\infty} \frac{1}{|a_j|^t} < \infty \right\}. \]

Remark 1. Last time, we saw if \( f(0) = 1 \),

\[ N(R) < \log M(3R), \quad N(R) \log 2 < \log M(2R) \]

Want: \( M \to \alpha \).

Can show: \( N \to \alpha \). In fact \( \alpha = \limsup_{R \to \infty} \frac{\log N(R)}{\log R} \) (we will not use).

Example \( a_n = n^2 \), \( N(R) = \left\{ j : |a_j| < R \right\} \sim R^{1/3} \).

\[ \alpha = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}} < \infty, \; 3t > 1, \; t > \frac{1}{3}. \text{Confirms } \alpha. \]

Recall Weierstrass factorization theorem (2208).

\[ f : \mathbb{C} \to \mathbb{C} \text{ entire with zeroes at } a_1, a_2, \ldots, |a_j| \neq 0, \ldots \]

\[ f(z) = z^m \prod_{n \geq 1} E_{p_n} \left( \frac{z}{a_n} \right) \text{ locally absolutely uniformly}. \]

When

\[ E_p(z) = \left\{ \begin{array}{ll}
1 - z & \text{if } p = 0 \\
(1 - z)^{\exp \left( z + \frac{z^2}{2} + \ldots + \frac{z^p}{p} \right)} & \text{if } p > 0.
\end{array} \right. \]

where \( \sum_{n \geq 1} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty \quad r. \text{This is always possible } p_n = n. \)

We need further restrictions.
Karl Weierstrass (1815 – 1897) was a German mathematician often cited as the “father of modern analysis”.

Despite leaving university without a degree, he studied mathematics and trained as a teacher, eventually teaching mathematics, physics, botany and gymnastics.

Factorization Theorem (1876) \( f : \mathbb{C} \rightarrow \mathbb{C} \) entire can be represented as

\[
f(z) = z^m \prod_{n} E_{p_n}\left(\frac{a_n}{z}\right)\]

where the elementary factors are

\[
E_p(z) = \begin{cases} 
1 - \frac{2}{z} & \text{if } p = 0 \\
(1 - \frac{2}{z}) \exp\left(\frac{z^2}{2} + \frac{z^4}{4} + \cdots + \frac{z^{2p}}{p!}\right) & \text{if } p > 0 
\end{cases}
\]
Assume \( g \) polynomial (growth) of degree \( q \)

\[ \exists \ p \ \text{integer, } \sum_{n=1}^{\infty} \frac{1}{\log(n)^p} < \infty. \tag{**} \]

The smallest \( p \) is called rank.

\[ \Rightarrow \ p+1 \geq \alpha \geq p. \]

**Example**

\[ f(x) = \sum_{n=1}^{\infty} \left( 1 - x^n \right) \text{ if } |x| < 1. \quad \text{Genus } \gamma \]

\[ = \sum_{n=1}^{\infty} x^n \left( \frac{x}{2^x} \right). \]

\[ p=0 \text{ works in (**) }, \sum_{n=1}^{\infty} \left( \frac{x}{2^x} \right)^n = \sum_{n=1}^{\infty} 2^n < \infty \]

\[ p=0, \quad q = -\infty, \quad \gamma = 0 \]

\[ \sin x = \sum_{n=1}^{\infty} \left( 1 - x^{2n} \right). \quad \text{Genus } \gamma \]

\[ = \sum_{n=1}^{\infty} \left( \frac{x}{n^2 \pi^2} \right)^n \exp \left( \frac{x}{n \pi} \right) \exp \left( -\frac{x}{n \pi} \right) \]

\[ = \sum_{n=1}^{\infty} \left( \frac{x}{n \pi} \right)^n \Rightarrow \ p=1 \Rightarrow \gamma = 1. \]

\[ p=1 \text{ works } \sum \frac{1}{(n \pi)^2} < \infty \text{ in (**) } \gamma \]

**Question A**

Knowing order \( \alpha \) \( \Rightarrow \) what is the genus \( \gamma \)?

**Theorem (Hadamard)** \( \gamma \leq \alpha \leq \gamma + 1. \quad \alpha \in \mathbb{Z} \)

**Question B**

Knowing the genus \( \gamma \) \( \Rightarrow \) what can we say about the order?

**Fact** \( \quad \text{order } \sum_{n=1}^{\infty} \exp \left( \frac{x}{a_n} \right) = \alpha = \text{critical exponent}. \quad (\text{We will not use}). \)
Jacques Hadamard  
1865 - 1963 (age 97)  
Proved the prime number theorem

Institutions

University of Bordeaux  
Sorbonne  
College de France  
École Polytechnique  
École Centrale Paris

Doctoral advisor

Émile Picard

Doctoral students

Maurice Fréchet  
Marc Krasner  
Paul Lévy  
André Weil

\[ f(z) = \sum_{n=1}^{\infty} E_p \left( \frac{z}{a_n} \right) \]

where \( g \) polynomial of degree \( \leq \gamma \), \( p = [a] \).

for entire functions of order \( \gamma \), genus \( h \)

\( h \leq \gamma \leq h + 1 \).
Proof that $\lambda \leq \frac{p+1}{2}$.

Lemma: $\log |E_p(w)| \leq C_p |w|^{p+2}$ for some constant $C_p > 0$.

Proof: $p = 0$: $\log |1 - w| \leq \log (1 + |w|) \leq |w|$. We induct on $p$, $p-1 \rightarrow p$. We have two cases.

2. If $|w| \geq \frac{1}{2}$: $|E_p(w)| = |E_{p-1}(w)| |\exp \left( \frac{w^p}{p} \right)|$

\[
\log |E_p(w)| = \log |E_{p-1}(w)| + \log |\exp \left( \frac{w^p}{p} \right)|
\]

Induction

\[
\leq C_{p-1} |w|^{p+1} + \log |\exp \Re \left( \frac{w^p}{p} \right)|
\]

\[
= C_{p-1} |w|^{p+1} + \Re \left( \frac{w^p}{p} \right)
\]

\[
\leq C_{p-1} |w|^{p+1} + \frac{1}{p} |w|^p = \left( C_{p-1} + \frac{1}{p} \right) |w|^{p+1}
\]

\[
\leq 2 \left( C_{p-1} + \frac{1}{p} \right) |w|^{p+1}
\]

\[
\leq \frac{C_p}{C_p}. \quad \text{Take } C_p > 2 \left( C_{p-1} + \frac{1}{p} \right).
\]

3. If $|w| \leq \frac{1}{2}$: $E_p(w) = \left( 1 - w \right) \exp \left( w + \frac{w^2}{2} + \frac{w^3}{3} + \ldots + \frac{w^p}{p} \right)$.

\[
\log |E_p(w)| = \Re \left( - \frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \ldots \right)
\]

\[
\leq \left| \frac{w^{p+1}}{p+1} \right| + \left| \frac{w^{p+2}}{p+2} \right| + \ldots
\]

\[
\leq |w|^{p+1} + |w|^{p+2} + \ldots
\]

\[
= |w|^{p+1} \left( 1 + |w| + \ldots \right)
\]

\[
\leq |w|^{p+1} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots \right) = 2 |w|^{p+2}.
\]

\[
\text{Take } c_p = \max \left( 2, 2 \left( C_{p-1} + \frac{1}{p} \right) \right).
\]

\[
\text{Take } C_p > 2.
\]
Proof that \( \lambda \leq h+1 \)

We know \( T = 2^m \in \{ \frac{2}{a_n} \}_{n \geq 1} \).

\( \lambda(2^m) = 0 \)

\( \lambda(2^m) = \deg g = q \leq h+1 \)

\( \lambda(u^q) \leq \max (2\lambda, 2\lambda^q) \)

Suffices to show \( \prod_{n \geq 1} E_p \left( \frac{2}{a_n} \right) \) has order \( \leq p+1 \leq h+1 \).

We estimate

\[ \log \left( \prod_{n \geq 1} E_p \left( \frac{2}{a_n} \right) \right) = \sum_{n \geq 1} \log \left( E_p \left( \frac{2}{a_n} \right) \right) \leq \]

Lemma

\[ \leq \sum_{n \geq 1} c_p \left| \frac{2}{a_n} \right|^{p+1} = C \left( 2 \right)^{p+1} \]

where \( C = c_p \sum_{n \geq 1} \frac{1}{|a_n|^{p+1}} < \infty. \)

Thus

\[ \log M(R) \leq CR^{p+1}, \]

where \( M(z) \) is computed for (\#)

\[ \log \log M(R) \leq \log C + (p+1) \log R. \]

\[ \implies \lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R} \leq \lim_{R \to \infty} \frac{\log C + (p+1) \log R}{\log R} = p+1. \]

\[ \boxed{\text{QED Q.}} \]

The inequality \( \lambda \geq h \) will be proved next time.