Solutions: Homework 1

Problem 1. Show that the function $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$u(z) = \log |z|$$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$.

Proof. For $(x, y) \neq (0, 0)$, we have $u(x + iy) = \frac{1}{2} \log(x^2 + y^2)$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and hence $u$ is harmonic.

Now, suppose that there exists $v : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ such that $f = u + iv$ is analytic on $\mathbb{C} \setminus \{0\}$. Let $U = \mathbb{C} \setminus \{z : \text{Re } z \leq 0\}$ and let $\text{Log}$ denote the principal branch of the logarithm. Then $\text{Re}(f|_U - \text{Log}) = 0$, hence $f|_U - \text{Log}$ has image contained in the imaginary line. By the Open Mapping Theorem, we must have that $f|_U - \text{Log}$ is constant. This implies that $\text{Log}$ has a continuous extension to $\mathbb{C} \setminus \{0\}$. This is a contradiction, completing the proof.

Problem 2. Let $G \subset \mathbb{C}$ be a symmetric region with respect to the real axis, and let

$$G^+ = G \cap \{\text{Im } z > 0\}$$

be the part in the upper half plane. Moreover, assume that $u$ is harmonic on $G^+$ and that

$$\lim_{z \to z_0} u(z) = 0$$

for any point $z_0 \in G \cap \mathbb{R}$. Show that $u$ extends to a harmonic function on $G$, and the extension satisfies

$$u(\overline{z}) = -u(z).$$
Proof. Extend \( u \) to \( G \) by defining \( u(z) = 0 \) for \( z \in G \cap \mathbb{R} \) and \( u(z) = -u(z) \) for \( z \in G^- \). Clearly \( u \) is continuous on \( G \).

We claim \( u \) is harmonic. Clearly \( u \) is harmonic in \( G^+ \). Furthermore, \( u \) is harmonic on \( G^- \) because for any \( (x_0, y_0) \in G^- \) we have
\[
\begin{align*}
    u_x(x_0, y_0) &= -u_x(x_0, -y_0),
    u_y(x_0, y_0) = u_y(x_0, -y_0),
    u_{xx}(x_0, y_0) &= -u_{xx}(x_0, -y_0),
    u_{yy}(x_0, y_0) = -u_{yy}(x_0, -y_0)
\end{align*}
\]
and thus
\[
u_{xx} + u_{yy} = 0\]
on \( G^- \) from the Laplace equation in \( G^+ \). Finally, let \( \alpha \in G \cap \mathbb{R} \). Let \( 0 < r < R \). Then
\[
\frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})d\theta = \frac{1}{2\pi} \int_0^{\pi} u(a + re^{i\theta})d\theta + \frac{1}{2\pi} \int_{\pi}^{2\pi} u(a + re^{i\theta})d\theta
\]
\[
= \frac{1}{2\pi} \int_0^{\pi} u(a + re^{i\theta})d\theta - \frac{1}{2\pi} \int_{\pi}^{2\pi} u(a + re^{-i\theta})d\theta = 0 = u(a).
\]
By the Mean Value Property, \( u \) is harmonic around any point on \( G \cap \mathbb{R} \), hence harmonic in \( G \).

\[\square\]

**Problem 3.** Let \( u : G \to \mathbb{R} \) be a nonconstant harmonic function in a region \( G \subset \mathbb{C} \). Show that \( u \) is an open map.

**Proof.** Since open sets are unions of open balls, it suffices to check that images of open balls are open. Let \( \Delta(a, r) \subset G \). Since \( \Delta(a, r) \) is simply connected, there exists \( v : \Delta(a, r) \to \mathbb{R} \) such that \( f = u + iv \) defined on \( \Delta(a, r) \) is analytic. Since \( f \) is analytic, \( f(\Delta(a, r)) \) is open in \( \mathbb{C} \). Note that \( \operatorname{Re} : \mathbb{C} \to \mathbb{R} \) given by \( \operatorname{Re}(x + iy) = x \) is an open map. Note that \( u(\Delta(a, r)) = \operatorname{Re}(f(\Delta(a, r))) \), and hence is open. So \( u \) is an open map. \( \square \)

**Problem 4.** Show that if \( u : \mathbb{R}^2 \to \mathbb{R} \) is harmonic and \( u(z) \geq 0 \) then \( u \) is constant.

**Proof.** Since \( \mathbb{C} \) is simply connected, there exists \( f \) entire such that \( \operatorname{Re} f = u \). Let \( g = e^{-f} \). Then \( g \) is also entire, and for all \( z \in \mathbb{C} \), \( |g(z)| = e^{-u(z)} \leq 1 \). By Liouville’s theorem, \( g \) is constant, and hence so is \( f \). This implies that \( u \) is constant. \( \square \)
Problem 5. Let $u : G \to \mathbb{R}$ be harmonic, and $\Delta(a, r) \subset G$. Let

$$M = \sup_{|z-a|=r} |u(z)|.$$

(i) Show that

$$u(a) = \frac{1}{\pi r^2} \int \int_{\Delta(a, r)} u(x, y) dxdy.$$

(ii) Show that the derivatives $u_x$ and $u_y$ are also harmonic. Therefore

$$u_x(a) = \frac{1}{\pi r^2} \int \int_{\Delta(a, r)} u_x(x, y) dxdy.$$

(iii) Use Green’s theorem in part (ii) and deduce that

$$|u_x(a)| \leq \frac{2}{r} M.$$

Derive the similar statement for $u_y$.

(iv) Using induction, show that for $i + j = n$ then the higher derivatives satisfy the estimates

$$|\partial^i_x \partial^j_y u(a)| \leq C_n r^{-n} M$$

for some constant $C_n$.

(v) Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and $|u(z)| \leq A(1 + |z|^m)$ then $u$ is a polynomial.

Proof. (i) Changing into polar coordinates, we have

$$\frac{1}{\pi r^2} \int \int_{\Delta(a, r)} u(x + iy) dxdy = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(a + \rho e^{i\theta}) \rho d\theta d\rho$$

Since $u$ is harmonic, for any $0 < \rho < r$, we have

$$\int_0^{2\pi} u(a + \rho e^{i\theta}) d\theta = 2\pi u(a)$$

Hence we have

$$\frac{1}{\pi r^2} \int \int_{\Delta(a, r)} u(x + iy) dxdy = \frac{2u(a)}{r^2} \int_0^r \rho d\rho = u(a)$$

(ii) Since $u$ is harmonic, $u$ is infinitely differentiable. So,

$$(u_x)_{yy} = (u_{yy})_x = (-u_{xx})_x = -(u_x)_{xx}.$$
Hence
\[
(u_x)_{xx} + (u_x)_{yy} = 0
\]
and \(u_x\) is therefore harmonic. Similarly, \(u_y\) is also harmonic. So, by part (i)
\[
u_x(a) = \frac{1}{\pi r^2} \int \int_{\Delta(a,r)} u_x(x,y) \, dx \, dy
\]
(iii) By Green’s theorem, we have
\[
u_x(a) = \frac{1}{\pi r^2} \int \int_{\Delta(a,r)} u_x(x,y) \, dx \, dy = \frac{1}{\pi r^2} \int_{\partial \Delta(a,r)} u(x,y) \, dy
\]
Then, changing \(x\) to \(\text{Re}(a) + r \cos \theta\) and \(y\) to \(\text{Im}(a) + r \sin \theta\), we have
\[
u_x(a) = \frac{1}{\pi r^2} \int_0^{2\pi} u(a + re^{i\theta}) r \cos \theta d\theta = \frac{1}{\pi r} \int_0^{2\pi} u(a + re^{i\theta}) \cos \theta d\theta
\]
Hence, we have
\[
|\nu_x(a)| \leq \frac{M}{\pi r} \int_0^{2\pi} |\cos \theta| d\theta = \frac{4M}{\pi r} \leq \frac{2rM}{r}.
\]
Similarly, we have
\[
u_y(a) = \frac{1}{\pi r^2} \int \int_{\Delta(a,r)} u_y(x,y) \, dx \, dy = -\frac{1}{\pi r^2} \int_{\partial \Delta(a,r)} u(x,y) \, dx
\]
Then, changing \(x\) to \(\text{Re}(a) + r \cos \theta\) and \(y\) to \(\text{Im}(a) + r \sin \theta\), we have
\[
u_y(a) = \frac{1}{\pi r^2} \int_0^{2\pi} u(a + re^{i\theta}) r \sin \theta d\theta = \frac{1}{\pi r} \int_0^{2\pi} u(a + re^{i\theta}) \sin \theta d\theta
\]
Hence, we have
\[
|\nu_y(a)| \leq \frac{M}{\pi r} \int_0^{2\pi} |\sin \theta| d\theta = \frac{4M}{\pi r} \leq \frac{2rM}{r}.
\]
(iv) We proceed by induction on \(n\). It is true for \(n = 1\) with \(C_1 = 2\). Assume that the inequalities are true for \(n - 1\). Since \(u_{xx} = -u_{yy}\) on \(G\), if \(j\) is even, we have
\[
|\partial_x^j \partial_y^j u(a)| = |\partial_x^n u(a)|
\]
and if \(j\) is odd, we have
\[
|\partial_x^j \partial_y^j u(a)| = |\partial_x^{n-1} u_y(a)|.
\]
In either case, we have, by our induction hypothesis applied to \( r/2 \) instead of \( r \):
\[
|\partial^i_x \partial^j_y u(a)| = |\partial^{n-1}_x v(a)| \leq \frac{C_{n-1}}{(r/2)^{n-1}} \sup_{|z-a|=r/2} |v(z)|
\]
where \( v = u_x \) if \( j \) is even and \( = u_y \) if \( j \) is odd. In either case, by (iii) above, we have for any \( z_0 \in \partial \Delta(a, r/2) \),
\[
|v(z_0)| \leq \frac{2}{r/2} \sup_{|z-z_0|=r/2} |u(z)|
\]
Since for any \( z_0 \in \partial \Delta(a, r/2) \), we have \( \partial \Delta(z_0, r/2) \subset \Delta(a, r) \), by the Maximum principle, we obtain that
\[
|v(z_0)| \leq \frac{4M}{r}
\]
and hence
\[
\sup_{|z-a|=r/2} |v(z)| \leq \frac{4M}{r}.
\]
Putting this back into the first inequality above, we get
\[
|\partial^i_x \partial^j_y u(a)| \leq \frac{2^{n+1}C_{n-1}}{r^n} M
\]
Putting \( C_n = 2^{n+1}C_{n-1} \), we are done.

(v) Let \( a \in \mathbb{R}^2 \). For any \( r > 0 \), we have
\[
\sup_{|z-a|=r} |u(z)| \leq A(1 + (|a| + r)^m)
\]
So, by part (iv) above, we have for \( i + j > m \),
\[
|\partial^i_x \partial^j_y u(0)| \leq \frac{AC_n}{r^{i+j}} (1 + (|a| + r)^m)
\]
Letting \( r \to \infty \), we have
\[
\partial^i_x \partial^j_y u(a) = 0
\]
for all \( a \in \mathbb{R}^2 \), whenever \( i + j > m \). In particular,
\[
\partial^{m+1}_x u \equiv 0.
\]
This implies that
\[
u = x^m f_m(y) + x^{m-1} f_{m-1}(y) + \ldots + f_0(y)
\]
for some $f_i$'s from $\mathbb{R} \to \mathbb{R}$. Similarly, $\partial_y^{m+1} u \equiv 0$ and hence the $f_i$'s are polynomials in $y$ of degree $\leq m$. This shows that $u$ is a polynomial.

\[ \square \]

**Problem 6.** If $u : \mathbb{R}^2 \to \mathbb{R}$ is bounded and harmonic then $u$ is constant.

**Proof.** Let $M$ be such that $u(z) \leq M$ for all $z \in \mathbb{C}$. Then $v := M - u$ is also harmonic. By Problem 4, $v$, and hence $u$, is constant. \[ \square \]

**Problem 7.** Show that if $u : \Delta(0,1) \setminus \{0\} \to \mathbb{R}$ is harmonic and $\lim_{z \to 0} u(z)$ exists, then $u$ can be extended to a harmonic function on $\Delta$.

**Proof.** Suppose $u : \Delta(0,1) \setminus \{0\} \to \mathbb{R}$ is harmonic. Then

$$\int_0^{2\pi} u(re^{it}) dt = a \log r + b$$

for some constants $a, b$ and for $0 < r < 1$. For a proof of this, see Theorem 3.6, VIII, Section 3 (Pg. 263) in “Complex Analysis”, Fourth Edition by S. Lang. Extend $u$ to $\Delta(0,1)$ by defining $u(0) = \lim_{z \to 0} u(z)$. Then $u$ is continuous on $\Delta(0,1)$. For $\epsilon > 0$ arbitrary, there exists $r_\epsilon > 0$ such that $|u(z) - u(0)| \leq \epsilon$ for all $|z| < r_\epsilon$. Then,

$$|a \log r + b - 2\pi u(0)| = \left| \int_0^{2\pi} (u(re^{it}) - u(0)) dt \right| \leq 2\pi \epsilon$$

for all $0 < r < r_\epsilon$. This can happen only if $a = 0$ and $b = 2\pi u(0)$. So we have

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it}) dt$$

for all $r > 0$. Hence, $u$ is harmonic on $\Delta$. \[ \square \]