Solutions: Homework 6

Problem 1. If \( p \neq 0 \) is a polynomial and \( a \neq 0 \), show that \( p(z) - e^{az} \) has infinitely many zeros.

Proof. Let

\[
 f(z) = p(z) - e^{az}.
\]

Then, the order of \( f \) is \( \leq \max\{\text{order of } p, \text{order of } e^{az}\} \). Since \( p \) is a polynomial, it has order 0 and the order of \( e^{az} \) is obviously 1. So, \( f \) has finite order \( \leq 1 \). So, by Hadamard’s factorization theorem, the genus of \( f \) is \( \leq 1 \). Suppose that \( f \) has only finitely many non-zero zeros

\[
a_1, \ldots, a_n
\]

counted according to multiplicity. Therefore

\[
 f = cz^m \left( 1 - \frac{z}{a_1} \right) \ldots \left( 1 - \frac{z}{a_n} \right) e^{az}
\]

for some \( \alpha \). In particular,

\[
 f = ge^{\alpha z} \iff p - e^{az} = ge^{az}
\]

for some polynomial \( g \). Now, let \( d = \deg f + 1 \). Differentiating \( d \) times we obtain

\[
 -a^d e^{az} = g_d e^{az}
\]

for some polynomial \( g_d \). This shows that \( a = \alpha \), and hence \( e^{az} = \frac{p}{g_d} \), a rational function, which is not possible. This contradicts our assumption that \( f \) has only finitely many zeros.

\[
 \square
\]

Problem 2. Give an example of a harmonic function on the unit disc \( u : \Delta \rightarrow \mathbb{R} \) such that

\[
 \lim_{r \to 1} u(re^{it}) = \begin{cases} 
 1 & \text{if } 0 < t < \pi, \\
 -1 & \text{if } \pi < t < 2\pi.
\end{cases}
\]

Proof. Let \( G = \{ z : \text{Re } z > 0 \} \). Let \( \partial G^+ = \{ z : \text{Re } z = 0, \text{Im } z > 0 \} \) and \( \partial G^- = \{ z : \text{Re } z = 0, \text{Im } z < 0 \} \). We first note that the function \( h : G \rightarrow \mathbb{R} \) given by

\[
 h(z) = \frac{2}{\pi} \text{Im}(\log z)
\]

satisfies the desired properties.
where \(\log\) denotes the principal branch of the logarithm, is obviously harmonic. Note that we also have \(h(z) \to 1\) as \(z \to z_0\) for any \(z_0 \in \partial G^+\) and \(h(z) \to -1\) as \(z \to z_0\) for any \(z_0 \in \partial G^-\). Now, the Möbius transformation \(\phi\) given by
\[
\phi(z) = \frac{1 + z}{1 - z}
\]
is analytic and sends \(\Delta \to G\), \(\partial \Delta \cap \{z : \text{Im } z > 0\} \to \partial G^+\) and \(\partial \Delta \cap \{z : \text{Im } z < 0\} \to \partial G^-\). In particular, the function \(u = h \circ \phi\) is harmonic on \(\Delta\) and a simple calculation gives us that
\[
u(re^{it}) = \frac{2}{\pi} \arctan \left( \frac{2r \sin t}{1 - r^2} \right)
\]
It is then easy to see that
\[
\lim_{r \to 1} \nu(re^{it}) = 1 \quad \text{if } 0 < t < \pi, \quad \lim_{r \to 1} \nu(re^{it}) = -1 \quad \text{if } \pi < t < 2\pi.
\]

**Problem 3.** Let \(f\) be holomorphic in the unit disc \(\Delta\) with \(f(z) \neq 0\) for \(z \neq 0\), and let
\[
M(r) = \sup_{|z|=r} |f(z)|
\]
for \(r < 1\). Assume
\[
M \left( \frac{1}{2} \right) M \left( \frac{1}{8} \right) = M \left( \frac{1}{4} \right)^2.
\]
Show that \(f(z) = az^n\).

**Proof.** Let \(n\) be the order of zero of \(f\) at 0. Then \(f(z) = z^n g(z)\) for some \(g\) holomorphic in \(\Delta\) with \(g(z) \neq 0\) for all \(z \in \Delta\). Let
\[
\widetilde{M}(r) = \sup_{|z|=r} |g(z)| = \sup_{|z|=r} \frac{|f(z)|}{|z|^n} = \frac{M(r)}{r^n}
\]
Then, we have
\[
\widetilde{M} \left( \frac{1}{2} \right) \widetilde{M} \left( \frac{1}{8} \right) = \frac{M \left( \frac{1}{2} \right)}{(1/2)^n} \frac{M \left( \frac{1}{8} \right)}{(1/8)^n} = \left( \frac{M \left( \frac{1}{4} \right)}{(1/4)^n} \right)^2 = \widetilde{M} \left( \frac{1}{4} \right)^2
\]
Now, let
\[
s = \frac{\log \widetilde{M} \left( \frac{1}{2} \right) - \log \widetilde{M} \left( \frac{1}{8} \right)}{-2 \log 2}
\]
and let \( h : \Delta \setminus \{0\} \to \mathbb{R} \) be defined by
\[
h(z) = s \log |z| + \log |g(z)|
\]
Since \( \log |z| \) is harmonic on \( \mathbb{C} \setminus \{0\} \) and \( g \) is analytic, we see that \( h \) is harmonic. Now, if \( |z| = \frac{1}{8} \), we have
\[
h(z) = -3s \log 2 + \log |g(z)|
\]
Then,
\[
\max_{|z| = \frac{1}{8}} h(z) = -3s \log 2 + \log \tilde{M}\left(\frac{1}{8}\right) - \frac{1}{2} \log \tilde{M}\left(\frac{1}{2}\right)
\]
Similarly, we have
\[
\max_{|z| = \frac{1}{4}} h(z) = -2s \log 2 + \log \tilde{M}\left(\frac{1}{4}\right) - \frac{1}{2} \log \tilde{M}\left(\frac{1}{2}\right)
\]
and
\[
\max_{|z| = \frac{1}{2}} h(z) = -s \log 2 + \log \tilde{M}\left(\frac{1}{2}\right) - \frac{1}{2} \log \tilde{M}\left(\frac{1}{8}\right)
\]
So we have
\[
\max_{|z|=\frac{1}{8}} h(z) = \max_{|z|=\frac{1}{4}} h(z) = \max_{|z|=\frac{1}{2}} h(z)
\]
Now, let \( G = \{z : 1/8 < |z| < 1/2\} \). Since \( h \) is harmonic and continuous on \( \overline{G} \), which is a compact set, \( h \) attains its maximum somewhere in \( \overline{G} \), say at \( a \). If \( |a| \neq 1/8 \) or \( 1/2 \), then by the Maximum Principle for harmonic functions, we get that \( h \) is constant on \( G \). If \( |a| = 1/8 \) or \( 1/2 \), then by what we deduced above, there exists \( b \) with \( |b| = 1/4 \) with \( h(b) \geq h(z) \) for all \( z \in G \). In either case, \( h \) is constant on \( G \). In particular, on \( G \setminus (-1/2, -1/8) \), the analytic function \( g(z)z^s \) is constant. This shows that
\[
g(z) = az^{-s}
\]
on \( \Delta \) for some \( a \in \mathbb{C} \), and since \( g(0) \neq 0 \), we have \( s = 0 \), and hence \( f(z) = az^n \). \( \square \)
**Problem 4.** Show that if $f$ is a meromorphic function satisfying a monic equation

$$f^n + a_1 f^{n-1} + \ldots + a_n = 0$$

whose coefficients $a_i$ are entire of order less than or equal to $\lambda$, the same is true about $f$. That is, the entire functions of order $\leq \lambda$ are integrally closed in the field of meromorphic functions.

**Proof.** Suppose that $a$ is a pole of $f$ of order $d$. Then $a$ is a pole of $f^i$ of order $id$ for $0 \leq i \leq n$. So, we have

$$0 \neq \lim_{z \to a} (z - a)^n f(z) = - \lim_{z \to a} \sum_{i=0}^{n-1} a_{n-i}(z) f^i(z)(z - a)^{id}(z - a)^{(n-i)d}$$

$$= - \sum_{i=0}^{n-1} \lim_{z \to a} a_{n-i}(z) f^i(z)(z - a)^{id} \lim_{z \to a} (z - a)^{(n-i)d} = 0,$$

a contradiction. So $f$ has no poles and is entire.

Let $\epsilon > 0$. There exists $R > 0$ such that for all $1 \leq i \leq n$ and for $|z| > R$, we have

$$|a_i(z)| \leq e|z|^{|\lambda + \epsilon|}$$

Now, suppose that $|f(z)| \geq 1$ and $|z| > R$. Then

$$|f(z)|^n \leq \sum_{i=0}^{n-1} |a_{n-i}(z)||f(z)|^i \leq \sum_{i=0}^{n-1} e|z|^{|\lambda + \epsilon|}|f(z)|^{n-1}$$

So, if $|f(z)| \geq 1$ and $|z| > R$, we have

$$|f(z)| \leq ne|z|^{|\lambda + \epsilon|}$$

This inequality holds trivially if $|f(z)| < 1$ and $|z| > R$. So, for all $|z| > R$, we have

$$|f(z)| \leq ne|z|^{|\lambda + \epsilon|}$$

Since $\epsilon > 0$ is arbitrary, the order of $f$ is $\leq \lambda$. \hfill \Box

**Problem 5.** Let $f$ be entire such that $f(z) \neq 0$ and $f^{-1}(1)$ is finite. Prove that $f$ is constant.
Proof. Suppose $f$ is non-constant. Since $f(z) \neq 0$, $f$ cannot be a polynomial. So, by Corollary 4.4 to the Great Picard Theorem, $f$ should assume every non-zero complex number infinitely often, and hence $f^{-1}(1)$ has to be infinite, a contradiction. Hence $f$ is constant.

Problem 6. Find the order of the function

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n!}\right).$$

Proof. Fix $0 < \epsilon < 1$. There exists $C > 0$ such that

$$\log |1 - z| \leq |\log |1 - z|| \leq C|z|^\epsilon$$

for all $|z| \leq 1/2$. Now, for $z \neq n!$, we have $f(z) \neq 0$, and hence

$$\log |f(z)| = \sum_{n=1}^{\infty} \log \left|1 - \frac{z}{n!}\right|$$

Let $|z| = R$. Then

$$\log |f(z)| = \sum_{\frac{n}{R} > \frac{1}{2}} \log \left|1 - \frac{z}{n!}\right| + \sum_{\frac{n}{R} \leq \frac{1}{2}} \log \left|1 - \frac{z}{n!}\right|$$

$$\leq \sum_{n! \leq 2R} \log \left|1 - \frac{z}{n!}\right| + C \sum_{n! \geq 2R} \left|\frac{z}{n!}\right|^\epsilon$$

$$\leq \sum_{n! \leq 2R} \log \left|1 - \frac{z}{n!}\right| + CR^\epsilon \sum_{n=1}^{\infty} \frac{1}{(n!)^\epsilon}$$

Now, note that, for all $n \geq 1$, we have $n! \geq 2^{n-1}$ and so

$$\log |f(z)| \leq \sum_{n! \leq 2R} \log \left|1 - \frac{z}{n!}\right| + CR^\epsilon \sum_{n=1}^{\infty} \frac{2^\epsilon}{(2^\epsilon)^n}$$

By the triangle inequality applied to the first summand above, we have

$$\log |f(z)| \leq \sum_{n! \leq 2R} \log \left(1 + \frac{R}{n!}\right) + \left(\frac{2^\epsilon C}{2^\epsilon - 1}\right) R^\epsilon$$

5
\[ \leq \log(1 + R) \sum_{n^t < 2R} 1 + \left( \frac{2^e C}{2^e - 1} \right) R^e \]
\[ \leq \log(1 + R) \sum_{2^{n-1} < 2R} 1 + \left( \frac{2^e C}{2^e - 1} \right) R^e \]
\[ \leq \frac{\log(1 + R) \log(4R)}{\log 2} + \left( \frac{2^e C}{2^e - 1} \right) R^e \]

This shows that
\[ \log M(R) \leq \frac{\log(1 + R) \log(4R)}{\log 2} + \left( \frac{2^e C}{2^e - 1} \right) R^e \]

Then for \( R >> 0 \), we have
\[ \log M(R) \leq \left( \frac{2^e C}{2^e - 1} + \frac{1}{\log 2} \right) R^e \]

This shows that the order of \( f \) is \( \leq \epsilon \). Since \( \epsilon > 0 \) is arbitrary, the order of \( f \) is 0. \( \Box \)