Appendix to Lecture 8

April 14, 2021

Let G be bounded and assume each $a \in \partial G$ is a barrier. Let $f : \partial G \longrightarrow \mathbb{R}$ be continuous.

Theorem The Perron function u for (G, f) satisfies

$$\lim_{z \to a} u(z) = f(a)$$

Corollary The Perron function solves the Dirichlet Problem under the above assumptions.

We let ω be a barrier at a. Thus

- $\omega : \overline{G} \longrightarrow \mathbb{R}$, ω cont in $\overline{G}$, ω harmonic in G
- $\omega(a) = 0$, $\omega > 0$ in $\partial G \setminus \{a\}$.

Proof WLOG $f(a) = 0$. Let $\varepsilon > 0$. We show

[i] $$\limsup_{z \to a} u(z) \le \varepsilon$$

[ii] $$\liminf_{z \to a} u(z) \ge -\varepsilon$$

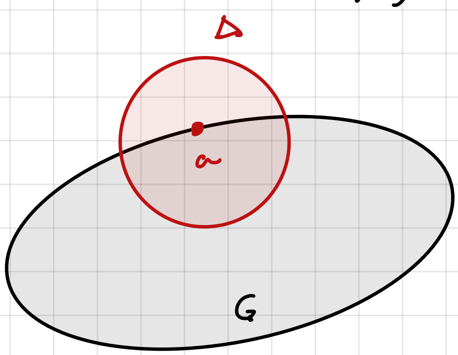
Then $\lim_{z \to a} u(z) = 0 = f(a)$, as needed.

Let Δ be a disc with

$$-\varepsilon < f < \varepsilon \quad \text{in } \partial G \cap \Delta \text{ and } a \in \Delta. \qquad (1)$$

Let $M = \sup_{\partial G} |f| \implies -M \le f \le M$ in ∂G. $\qquad (2)$

∂G = compact



Since $\overline{G} \setminus \Delta$ is compact, let

$$\omega_0 = \min_{\overline{G} \setminus \Delta} \omega > 0$$

Why? By Minimum Principle^{+}, either $\omega \equiv 0$ in G (not true as $\omega|_{\partial G} \not\equiv 0$) or else $\omega > 0$ in G. But $\omega > 0$ in $\partial G \setminus \{a\}$. Thus $\omega > 0$ in $\overline{G} \setminus \{a\}$. Since $\overline{G} \setminus \Delta \subseteq \overline{G} \setminus \{a\}$, we get the claim.

Proof of $\boxed{II}$

Let $\boxed{V(z) = -\varepsilon - \dfrac{\omega(z)}{\omega_0} \cdot M.}$ = harmonic in G.

cont in $\bar{G}$

Claim 1 $V \le f$ over ∂G

Proof Let $z \in \partial G$.

$\omega \ge 0$ on ∂G .

- $z \in \partial G \cap \Delta$: $V(z) \le -\varepsilon \overset{(1)}{<} f(z)$
- $z \in \partial G \setminus \Delta$: $V(z) < -M \overset{(2)}{\le} f(z)$

$\omega \ge \omega_0$ in $\bar{G} \setminus \Delta$

Claim 2 $V \in \mathcal{P}(G, f)$.

Proof We know V harmonic. For $\zeta \in \partial G$,

$$\lim_{z \to \zeta} V(z) = V(\zeta) \le f(\zeta) \text{ by Claim 1.}$$

Since u is defined as a supremum over $\mathcal{P}(G, f)$ & $V \in \mathcal{P}(G, f)$

$\Longrightarrow$ $u(z) \ge V(z)$ $\forall z \in G$

$\Longrightarrow$ $\liminf_{z \to a} u(z) \ge V(a) = -\varepsilon$ as needed.

$\hookrightarrow$ $\omega(a) = 0$.

Proof of $\boxed{\text{I}}$ Let

$$W(z) = \varepsilon + \frac{\omega(z)}{\omega_0} \cdot M = \text{harmonic in } G, \text{ cont. in } \bar{G}.$$

Claim 1' $W \geq f$ over ∂G.

$\omega \geq 0$ in ∂G

Proof

- $z \in \partial G \cap \Delta$, $W(z) \geq \varepsilon \overset{(1)}{>} f(z)$

- $z \in \partial G \setminus \Delta$, $W(z) > M \overset{(2)}{\geq} f(z)$

$\omega \geq \omega_0$ in $\bar{G} \setminus \Delta$

We do not know $W \in \mathcal{P}$, but we can compare W to any $\varphi \in \mathcal{P}$

Claim 2' $W(z) \geq \varphi(z)$ $\forall \varphi \in \mathcal{P}$ $\forall z \in G$.

Proof Let $\zeta \in \partial G$. Then

definition of $\mathcal{P}$ claim 1'

$$\limsup_{z \to \zeta} \varphi(z) \leq f(\zeta) < W(\zeta) = \lim_{z \to \zeta} W(z)$$

$\Rightarrow \varphi(z) \leq W(z)$ $\forall z \in G$ by MP^+ applied to the function $\varphi - W$.

Since $u(z) = \sup\{\varphi(z) : \varphi \in \mathcal{P}\} \Rightarrow u(z) \leq W(z)$ by

Claim 2 $\forall z \in G$. Then

$$\limsup_{z \to a} u(z) \leq \lim_{z \to a} W(z) = W(a) = \varepsilon,$$ as needed.