# Math 220 C - Lecture 1

March 29, 2021

## 0 Logistics

(1) Zoom lectures — MWF 3 – 3:50 PM.

2nd half — TBD.

(2) Office Hour — W 4 – 5:30 PM

(3) PSets — due Fridays, weekly

(4) Grades — HWK & Attendance

(5) Qualifying Exam — TBA

(6) Canvas / Gradescope / Website

math.ucsd.edu/~doprea/220s21.html

(7) Attendance

## Topics for Math 220C

(1) Harmonic Functions — Conway X.

(2) Hadamard Factorization — Conway XI.

(3) Picard's theorems — Conway XII.

## Math "220D" (if time)

(4) Introduction to Riemann Surfaces.

# $\boxed{1.}$ Harmonic Functions

**Theme:**

Harmonic functions share many properties with holomorphic functions

$\boxed{i}$ mean value property & Integral formulas

$\boxed{ii}$ maximum modulus principle

$\boxed{iii}$ convergence theorems

& others $\rightsquigarrow$ HWK 1.

"Cauchy" estimates, Liouville, Open Mapping Thm.

**Convention** $G \subseteq \mathbb{C}$ open & connected. We will assume this from now on.

**Recall** $G \subseteq \mathbb{C}$ open & connected

$u : G \longrightarrow \mathbb{R}$ harmonic iff $u \in C^2$ and

$u_{xx} + u_{yy} = 0$. (Laplace equation).

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**Recall** (Harmonic conjugates, Math 220A, Lecture 1).

If $f : U \longrightarrow \mathbb{C}$ holomorphic $\Rightarrow$ $u = \operatorname{Re} f$ harmonic.

$v = \operatorname{Im} f$ harmonic

$u, v$ are said to be harmonic conjugates provided

$f = u + iv$ is holomorphic.

(so that $u = \operatorname{Re} f$, $v = \operatorname{Im} f$). Note that $u, v$ satisfy

the Cauchy Riemann equations

$$u_x = v_y$$

$$u_y = -v_x.$$

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**Lemma** Let $G$ be simply connected.

Any $u : G \longrightarrow \mathbb{R}$ harmonic admits a harmonic conjugate $v$.

e.g. $f = u + iv$ = holomorphic, $u = \operatorname{Re} f$.

**Proof** Let $F = u_x - i u_y$.

**Claim** $F$ holomorphic

Indeed, $F$ is of class $C^1$ & satisfies CR equations.

$$(u_x)_x = (-u_y)_y \iff u_{xx} + u_{yy} = 0 \quad \text{true}$$

$$(u_x)_y = -(-u_y)_x \iff u_{xy} = u_{yx}. \quad \text{true}$$

$\Rightarrow$ $F$ holomorphic by Math 220, Lecture 2.

Since $G$ is simply connected, $F$ admits a primitive

$\Rightarrow$ $F = f'$ for $f$ holomorphic, $f = \alpha + i\beta$.

$$f' = \alpha_x + i\beta_x = F = u_x - i u_y$$

$\Rightarrow$ $\alpha_x = u_x$ $\Rightarrow$ $\alpha = u + c$.

$\Rightarrow$ $\beta_x = -u_y = -\alpha_y$ $\Rightarrow$ $\alpha_y = u_y$.

Replacing $f$ by $f - c$, we obtain $u = \operatorname{Re} f$ & $v = \operatorname{Im} f$ is the conjugate of $u$.

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Remark  Math 220A, HWK 2

**3.** Show that the function $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$u(z) = \log |z|$$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$.

Thus the Lemma above fails for $G$ not simply connected.

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## Corollary

$u$ harmonic $\Longrightarrow$ $u$ is of class $\mathcal{C}^\infty$.

## Proof

Indeed, the statement is local. Let $a \in G$. Let $\overline{\Delta}(a,r) \subseteq G$.

Since $\Delta(a,r)$ simply connected, $u = \operatorname{Re} f$, $f$ holomorphic in $\Delta(a,r)$.

A holomorphic function is $\infty$-many times complex differentiable

& thus $\infty$-many times real differentiable. (Math 220A, Lecture 1).

$\Rightarrow$ $u$ is $\mathcal{C}^\infty$.

# First Properties of Harmonic Functions

[i] mean value property (MVP)

[ii] maximum principle (MP)

[iii] Poisson integral formula (next time)

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**Def** $u : G \longrightarrow \mathbb{R}$ continuous satisfies MVP if

$\forall\ a \in G\ ,\ \bar{\Delta}(a, r) \subseteq G.$

$$\underbrace{u(a)}_{\text{value at center}} = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} u(a + re^{it})\, dt}_{\text{average values over the boundary.}}$$

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**Theorem** $u : G \longrightarrow \mathbb{R}$ harmonic $\Rightarrow$ $u$ satisfies MVP.

**Proof** Let $\overline{\Delta}(a,r) \subseteq \Delta(a,R) \subseteq G$ Write

$u = \operatorname{Re} f$, $f$ holomorphic in $\Delta(a,R)$.

Cauchy Integral Formula gives

$z = a + r e^{it}$

$dz = r i e^{it} dt$

$$f(a) = \frac{1}{2\pi i} \int_{\partial \Delta(a,r)} \frac{f(z)}{z-a}\, dz.$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} \cdot r i e^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{it})\, dt.$$

Take real part on both sides & conclude.

# Maximum Principle

$u : G \longrightarrow \mathbb{R}$, $u \in C^0(G)$ satisfies MVP. Assume

$\exists\ a \in G$, $u(a) \geq u(z)$ $\forall z \in G$. Then $u$ is constant.

**Proof** Let $\Omega = \{ z : u(z) = u(a) \} \subseteq G$.

(1) $\Omega \neq \emptyset$ because $a \in \Omega$.

(2) $\Omega$ is closed, since $u$ is continuous.

(3) $\Omega$ is open.

Then $G$ connected $\Rightarrow \Omega = G \Rightarrow u$ constant.

**Proof of (3)**

Let $z_0 \in \Omega$. Let $\bar{\Delta}(z_0, r) \subseteq G$. We show $\Delta(z_0, r) \subseteq \Omega$.

Let $w \in \Delta(z_0, r)$. $\Rightarrow \rho = |w - z_0|$. Write MVP for $\partial\Delta(z_0, \rho)$

$$u(a) = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{it})\, dt.$$

$$\iff \frac{1}{2\pi}\int_0^{2\pi} \left(u(z_0+\rho e^{it}) - u(a)\right) dt = 0 \implies$$

Let $f(t) = u(a) - u(z_0+\rho e^{it})$. By assumption, $f(t) \geq 0$ since $a$ is a maximum for $u$.

Using the Lemma, we have $f \equiv 0$. Since $|w - z_0| = \rho$, write

$$w = z_0 + \rho e^{it_0} \implies f(t_0) = u(a) - u(w) = 0 \implies u(a) = u(w)$$

$\implies w \in \Omega \implies \Delta(z_0, r) \subseteq \Omega \implies \Omega$ open.

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**Lemma** $f : [0, 2\pi] \longrightarrow \mathbb{R}$, $f \geq 0$ and $f$ continuous

$$\int_0^{2\pi} f(t)\,dt = 0 \implies f \equiv 0.$$

**Proof** If $f(t_0) > 0$, by continuity we can find $\delta > 0$ such that $f(t) > \frac{f(t_0)}{2}$ $\forall\, t \in (t_0 - \delta, t_0 + \delta) \cap [0, 2\pi]$.

Assume $t_0 \neq 0, 2\pi$ since the proof is similar in those cases.

Then $f \geq 0$ gives

$$0 = \int_0^{2\pi} f(t)\, dt \geq \int_{t_0-\delta}^{t_0+\delta} f(t)\, dt > \int_{t_0-\delta}^{t_0+\delta} \frac{f(t_0)}{2}\, dt = \delta f(t_0) > 0.$$

contradiction. Thus $f \equiv 0$.

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## Remark

$\boxed{\text{i}}$ $u$ harmonic $\Rightarrow$ $u$ satisfies maximum principle

$\boxed{\text{ii}}$ $u$ harmonic $\Rightarrow$ $-u$ harmonic

$\Rightarrow$ $-u$ satisfies maximum principle

$\Rightarrow$ $u$ satisfies minimum principle

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Georg Friedrich Bernhard Riemann

17 September 1826 – 20 July 1866

Eine harmonische Function u kann nicht in einem Punkt im Innern ein Minimum oder ein Maximum haben, wenn sie nicht uberall constant ist.

(A harmonic function u cannot have either a minimum or a maximum at an interior point unless it is constant.)

"Grundlagen fur eine allgemeine Theorie der Functionen einer veranderlichen complexen Grosse”

Dissertation Gottingen (1851)