Math 220C - Lecture 10

April 19, 2021
§0. Last time \( f: \mathbb{C} \rightarrow \mathbb{C} \) entire function

Main Question: Establish relationship between

\[
\{ \text{Growth of } f \} \leftrightarrow \{ \text{Distributions of zeros} \}
\]

Subquestion: How do we interpret the two sides mathematically?

§1. Left hand side

Order

Recall \( M(R) = \sup_{|z| \leq R} |f(z)| \). & we defined

\[
\lambda(f) = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}
\]

Intuitively, "\( f(z) \sim e^{121^\lambda} \)"
Question

How to prove a function $f$ has order $\lambda$?

We need to show two statements:

1. $\forall \epsilon > 0 \exists r$ such that $|f(x)| < \epsilon$ for $|x| > r$

This shows $\lambda(f) \leq \lambda$.

$$\lambda(f) = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R} \leq \lambda + \epsilon \Rightarrow \lambda(f) \leq \lambda$$

2. $\forall \epsilon > 0 \exists 2^n \to \infty$ with $|f(2^n)| > \epsilon$ for $n > 1/\lambda - 3$

This shows $\lambda(f) \geq \lambda$.

$$\lambda(f) = \limsup_{n \to \infty} \frac{\log \log M(R)}{\log R} \geq \limsup_{n \to \infty} \frac{\log \log |f(2^n)|}{\log |2^n|} \geq \lambda - \epsilon \Rightarrow \lambda(f) \geq \lambda.$$
§2. Right hand side & Distribution (growth) of zeroes

Assume \( f \) has zeroes at

\[ |a_1| \leq |a_2| \leq \ldots \leq |a_n| \leq \ldots \quad , \quad a_n \to 0 , \quad a_n \neq 0 \]

Several quantities attached to growth of zeroes:

\[ \text{rank} = p \]

The smallest integer \( p \) such that

\[ \sum_{n=1}^{\infty} \frac{1}{|a_n|^p} < \infty \]

If such a \( p \) doesn't exist, \( p = \infty \).

\[ \alpha = \inf \{ t > 0 : \sum_{n=1}^{\infty} \frac{1}{|a_n|^t} < \infty \} \]

may not be an integer.

By the homework

\[ p \quad \alpha \quad p+1 \]

divergent series \quad convergent series

Thus by definition \( p \leq \alpha \leq p+1 \).

If \( \alpha \notin \mathbb{N} \) then \( \alpha \) determines \( p \) uniquely.
$N(R) = \# \text{zeros in } \Delta(0,R) \text{ with multiplicity}$

**Fact** *(we will not use/prove)*

$$\alpha = \limsup_{R \to \infty} \frac{\log N(R)}{\log R}$$

**Example** *

Let $a_n = n^3$, $n \geq 0$, then

$$N(R) = \# \{ n : n^3 < R \} \sim R^{1/3} \implies \frac{\log N(R)}{\log R} \to \frac{1}{3}$$

Note

$$\sum \frac{1}{n^{3t}} < \infty \iff 3t > 1 \iff t > \frac{1}{3} \Rightarrow \alpha = \frac{1}{3}.$$ **Harmonic Series**

**Upshot**

We have defined the following quantities measuring growth/distribution of zeros:

$N(R)$, $\alpha$, $p$.

Note $N(R)$ determines $\alpha$, $\alpha$ determines $p$ if $\alpha \notin \mathbb{Z}$.

Best for us: $p$ (or $k$ to be defined next).
**Small variation — Genus of an entire function**

Let \( f \) have zeroes at \( a_1, a_2, \ldots, a_n, \ldots, a_k \neq 0 \).

where \( \{ a_n \} \) has rank \( p \).

\[
\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty
\]

Recall Weierstrass Factorization

\[
f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p \left( \frac{2}{a_n} \right).
\]

Recall

\[
E_p(z) = \begin{cases} 
1 - z, & p = 0 \\
(1 - z) \exp \left( z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right), & p > 0
\end{cases}
\]

Define

\[
\mathcal{h} = \text{genus of } f = \begin{cases} 
\max (p, g) & \text{if } g \text{ polynomial of degree } g \\
\infty & \text{if } g \text{ not polynomial or } p = \infty
\end{cases}
\]

If the exponential \( z^g \) doesn't appear then \( h = p \).

In general \( p \leq h \).
Example (Math 2208)

\[ \sin^2 x = 2 \sum_{n=1}^{\infty} \left( 1 - \frac{2}{n^2 \pi^2} \right) \] factorization of \( \sin^2 \).

Rewrite this as

\[ \sin^2 x = 2 \sum_{n=1}^{\infty} \left( 1 - \frac{2}{n^2 \pi^2} \right) e^{\frac{2}{n\pi}} \left( 1 + \frac{2}{n\pi} \right) e^{-\frac{2}{n\pi}} \]

\[ = 2 \sum_{n=1}^{\infty} E_1 \left( \frac{2}{n\pi} \right) E_1 \left( -\frac{2}{n\pi} \right) \]

\[ \Rightarrow g \text{ doesn't appear. Thus genus } h = p. \]

The zeroes are at \( n\pi, n \in \mathbb{Z} \). We want

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{n\pi} \right)^{p+1} < \infty \] \( \Rightarrow \ p+1 > 1 \] \( \Rightarrow \ p > 0 \). Thus the smallest \( p \) equals 1.

The genus of \( \sin^2 \) equals 1.
§ 3. Revising the Main Question (now made precise)

Establish relationship between

\[
\begin{align*}
\{ \text{Growth of } f \} & \quad \leftrightarrow \quad \{ \text{Growth of zeroes} \} \\
\downarrow & \\
\text{measured by } \lambda & \quad \leftrightarrow \quad \text{measured by } h = \text{genus}
\end{align*}
\]

Answer Theorem (Hadamard)

\[ h \leq \lambda \leq h + 1 \]

Remarks

1. If \( \lambda \neq 2 \) then \( \lambda \) determines \( h \) uniquely.

2. If \( e^s \) doesn’t appear then \( h = p \) so in this case,

\[ p \leq \lambda \leq p + 1 \]

3. We have \( p \leq h \leq \lambda \) so the order bounds the \( p \) in the Weierstrass Factorization. The statement that we can take \( p \leq \lambda \) is called Hadamard Factorization.
Conclusion 7 connections between

- $M(R)$ and $\lambda$ by definition $\lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}$
- $N(R)$, $\alpha$, $p$ as we saw above
- $\lambda$ and $\ell = \max(p, 2)$ via Hadamard $\ell \leq \lambda \leq \ell + 1$

Next - proof that $\lambda \leq \ell + 1$

- proof that $\ell \leq \ell$

- Applications
Jacques Hadamard
1865 - 1963 (age 97)

Proved the prime number theorem

Institutions
- University of Bordeaux
- Sorbonne
- College de France
- École Polytechnique
- École Centrale Paris

Doctoral advisor
- Émile Picard

Doctoral students
- Maurice Fréchet
- André Weil