Math 220C - Lecture 10

April 19, 2021

§0. Last time

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire function

Main Question Establish relationship between

$\{$ Growth of $f \}$ $\longleftrightarrow$ $\{$ Distributions of zeros $\}$

Subquestion: How do we interpret the two sides mathematically?

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§1. Left hand side

Order Recall $M(R) = \sup_{|z|=R} |f(z)|$. & we defined

$$\lambda(f) = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}$$

Intuitively, " $f(z) \sim e^{|z|^{\lambda}}$ ".

Question How to prove a function f has order λ ?

We need to show two statements:

$\boxed{I}$ $\forall\, \varepsilon > 0\ \exists\, r$ such that $|f(z)| < e^{|z|^{\lambda+\varepsilon}}$ $\forall\, |z| > r$

This shows $M(R) < e^{R^{\lambda+\varepsilon}}$ $\forall\, R > r$ &

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \le \lambda+\varepsilon \quad \forall\, \varepsilon. \overset{\varepsilon\to 0}{\Longrightarrow} \lambda(f) \le \lambda$$

$\boxed{II}$ $\forall\, \varepsilon > 0$ $\exists\, z_n \longrightarrow \infty$ with $|f(z_n)| > e^{|z_n|^{\lambda-\varepsilon}}$

This shows

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \ge \limsup_{n\to\infty} \frac{\log\log |f(z_n)|}{\log |z_n|} \ge \lambda-\varepsilon$$

$$\overset{\varepsilon\to 0}{\Longrightarrow} \lambda(f) \ge \lambda .$$

Properties

$\boxed{i}$ $\lambda(z^m) = 0$, $M(r) = r^m \Rightarrow \lambda = 0$.

$\boxed{ii}$ $\lambda(e^P) = \deg P$ for P = polynomial (check)

$\boxed{iii}$ $\lambda(fg) \leq \max(\lambda(f), \lambda(g))$ (HWK 4)

§2. Right hand side & Distribution (growth) of zeroes

Assume f has zeroes at

$$|a_1| \le |a_2| \le \dots \le |a_n| \le \dots \quad . \quad a_n \longrightarrow \infty \,, \quad a_n \ne 0$$

Several quantities attached to growth of zeroes:

$\boxed{\text{i}}$ rank $= p$

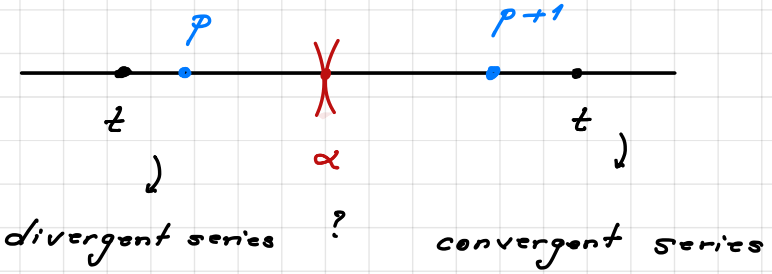
The smallest integer p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$.

If such a p doesn't exist, $p = \infty$.

$\boxed{\text{ii}}$ critical exponent (HWK 4, #5)

$\displaystyle\alpha = \inf\left\{ t > 0 : \sum \frac{1}{|a_n|^t} < \infty \right\}$ may not be an integer

By the homework



Thus by definition $p \le \alpha \le p+1$.

If $\alpha \notin \mathbb{Z}$ then α determines p uniquely.

[iii] $N(R)$ = # zeroes in $\Delta(0,R)$ with multiplicity

Fact* (we will not use/prove)

$$\alpha = \limsup_{R\to\infty} \frac{\log N(R)}{\log R}$$

Example* Let $a_n = n^3$, $n>0$. then

$$N(R) = \#\{n : n^3 < R\} \sim R^{1/3} \Rightarrow \frac{\log N(R)}{\log R} \longrightarrow \frac{1}{3}$$

Note

$$\sum \frac{1}{n^{3t}} < \infty \iff 3t > 1 \iff t > \frac{1}{3} \quad \text{so} \quad \alpha = \frac{1}{3}.$$

harmonic series

Upshot We have defined the following quantities measuring growth / distribution of zeroes

$N(R)$, α, p.

Note $N(R)$ determines α, α determines p if $\alpha \notin \mathbb{Z}$.

Best for us: p (or h to be defined next).

Small variation — Genus of an entire function

Let f has zeroes at $a_1, a_2 \ldots, a_n, \ldots, a_k \neq 0$.

where $\{a_n\}$ has rank p. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$

Recall Weierstraß Factorization

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right).$$

Recall

$$E_p(z) = \begin{cases} 1-z, & p=0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), & p>0 \end{cases}$$

Define

$$h = \text{genus}(f) = \begin{cases} \max(p, q) & \text{if } g \text{ polynomial of degree } q \\ \infty & \text{if } g \text{ not polynomial or } p = \infty. \end{cases}$$

If the exponential e^g doesn't appear then $h = p$.

In general $p \leq h$.

Example (Math 220B)

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$ factorization of sine.

Rewrite this as

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi}$$

$$= z \prod_{n=1}^{\infty} E_1\left(\frac{z}{n\pi}\right) E_1\left(-\frac{z}{n\pi}\right)$$

$\Rightarrow$ g doesn't appear. Thus genus $h = p$.

The zeroes are at $n\pi$, $n \in \mathbb{Z}$. We want

$$\sum_{n \neq 0} \frac{1}{|n\pi|^{p+1}} < \infty \iff p+1 > 1 \iff p > 0.$$ Thus the

harmonic series

smallest p equals 1.

The genus of $z \longmapsto \sin z$ equals 1.

§ 3. Revisiting the Main Question (now made precise)

Establish relationship between

$$\{\text{Growth of } f\} \longleftrightarrow \{\text{Growth of zeroes}\}$$

measured by λ — measured by h = genus.

Answer **Theorem** (Hadamard)

$$h \leq \lambda \leq h+1$$

Remarks $\boxed{i}$ If $\lambda \notin \mathbb{Z}$ then λ determines h uniquely.

$\boxed{ii}$ If e^g doesn't appear then $h = p$ so in this case.

$$p \leq \lambda \leq p+1$$

$\boxed{iii}$ We have $p \leq h \leq \lambda$ so the order bounds the p in the Weierstraß Factorization. The statement that we can take $p \leq \lambda$ is called Hadamard Factorization.

Conclusion $\exists$ connections between

- $M(R)$ and λ by definition $\lambda = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R}$
- $N(R)$, α, p as we saw above
- λ and $h = \max(p, q)$ via Hadamard $h \le \lambda \le h+1$

Next

- proof that $\lambda \le h+1$
- proof that $h \le \lambda$
- Applications

Jacques Hadamard (1865 - 1963)

Proved the Prime Number Theorem.

Advisor: Émile Picard.

Students: Maurice Fréchet, André Weil