Math 220C - Lecture 12

April 23, 2021

Theorem $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f \not\equiv 0$ entire. Then

$$h \leq \lambda \leq h+1.$$

Conway XI. 3

We already know $\lambda \leq h+1$. We show $h \leq \lambda$.

WLOG λ finite & $f(0) = 1$

Indeed, write $f(z) = c z^m \tilde{f}(z)$ with $\tilde{f}(0) = 1$. Note

$$\text{order } \tilde{f} = \text{order } f \quad \& \quad \text{genus } \tilde{f} = \text{genus } f.$$

WTS If λ is finite, then

$\boxed{i}$ $p \leq \lambda$

$\boxed{ii}$ g polynomial of degree $\leq \lambda$.

where $f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$.

Proof of $\boxed{\mathrm{I}}$ By HWK 4, Problem 5:

$\alpha \le \lambda$ and by Lecture 10, $\rho \le \alpha$. Thus $\rho \le \lambda$.

Proof of $\boxed{\mathrm{II}}$ is technical.

Let $m \le \lambda < m+1$.

Write $f(z) = e^{g(z)} P(z)$ where $P(z) = \prod_n E_p\left(\frac{z}{a_n}\right)$

We will prove

$$D^{m+1} g = 0 \quad \text{in } \mathbb{C} \setminus \{a_1, a_2 \ldots a_n \ldots\}.$$

This will show $D^{m+1} g = 0$ in $\mathbb{C}$, say by identity principle. $\Rightarrow$ g polynomial of degree $\le m$.

Here D = derivative $= \frac{\partial}{\partial z}$

Take logarithmic derivatives

$$f = e^{g} P \implies \frac{f'}{f} = g' + \frac{P'}{P}$$

Take m usual derivatives next to get

$$D^m \frac{f'}{f} = D^{m+1} g + D^m \frac{P'}{P}.$$

We will show $\boxed{D^m \frac{f'}{f} = D^m \frac{P'}{P}} \Rightarrow \boxed{D^{m+1} g = 0}$ as needed.

Claim $$D^m \frac{P'}{P} = -m! \sum_n \frac{1}{(a_n - z)^{m+1}}$$

Proof Recall $E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right).$

$$\Rightarrow \frac{E_p'(z)}{E_p(z)} = -\frac{1}{1-z} + 1 + z + \dots + z^{p-1}$$

$$\Rightarrow \boxed{D^m \frac{E_p'(z)}{E_p(z)} = -\frac{m!}{(1-z)^{m+1}}} + 0 \text{ since } p \le \lambda < m+1 \text{ by.}$$

Part $\boxed{4}$

Recall If $u = \prod_n u_n$

converges absolutely & locally uniformly then

$$\frac{u'}{u} = \sum_n \frac{u_n'}{u_n}$$

absolutely & locally uniformly away from zeroes. (Math 220B)

In our case

$$P(z) = \prod_n E_p\left(\frac{z}{a_n}\right)$$ converges absolutely & locally unif.

$$\Rightarrow \frac{P'}{P} = \sum_n \frac{\left(E_p\left(\frac{z}{a_n}\right)\right)'}{E_p\left(\frac{z}{a_n}\right)}$$

$$\Leftrightarrow D^m \frac{P'}{P} = \sum_n D^m \frac{\left(E_p\left(\frac{z}{a_n}\right)\right)'}{E_p\left(\frac{z}{a_n}\right)}$$

(switching differentiation & summation by Weierstraß convergence thm).

$$= -\sum_n \frac{m!}{(a_n - z)^{m+1}}$$ as needed.

Lemma f entire, $f(0)=1$, $m+1>\lambda$

$$D^m \frac{f'}{f} = -m! \sum_n \frac{1}{(a_n - z)^{m+1}}$$

The Lemma & above computation shows $D^m \frac{f'}{f} = D^m \frac{P'}{P}$ as claimed.

Proof By Poisson - Jensen formula in $\Delta(0,R)$, $z \neq a_k$:

$$\log|f(z)| + \sum_{k=1}^{N(R)} \log\left|\frac{R^2 - \bar{a}_k z}{R(z-a_k)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} \frac{Re^{it}+z}{Re^{it}-z} \cdot \log|f(Re^{it})|\,dt$$

Idea [i] differentiate

[ii] make $R \longrightarrow \infty$.

The Lemma will follow.

Remark $\quad 2\dfrac{\partial}{\partial z}\log|f| = \dfrac{\partial}{\partial z}\log|f(z)|^2$

$$= \frac{\partial}{\partial z}\log(f\cdot\bar{f})$$

$$= \frac{\partial}{\partial z}\log f + \frac{\partial}{\partial z}\log\bar{f}$$

$$= \frac{f'}{f} + \overline{\frac{\partial}{\partial\bar{z}}\log f}$$

$$= \frac{f'}{f} + 0.$$

This follows because $\log f$ is locally, away from zeroes, a holomorphic function and thus $\frac{\partial}{\partial\bar{z}}\log f = 0$ by Cauchy–Riemann equations (Math 220 A, Lecture 1).

Step 1 : Apply $2\frac{\partial}{\partial z}$ to Poisson-Jensen

$$\log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{R^2 - \bar{a}_k z}{R(z-a_k)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{Re^{it}+z}{Re^{it}-z} \cdot \log|f(Re^{it})|\,dt$$

Compute

$$\left(\frac{Re^{it}+z}{Re^{it}-z}\right)' = \left(-1 + \frac{2Re^{it}}{Re^{it}-z}\right)' = \frac{2Re^{it}}{(Re^{it}-z)^2}.$$

Differentiating, we obtain

$$\frac{f'}{f} = \sum_{k=1}^{N(R)} \frac{\bar{a}_k}{R^2 - \bar{a}_k z} - \sum_{k=1}^{N(R)} \frac{1}{a_k - z} + \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{2Re^{it}}{(Re^{it}-z)^2} \log|f(Re^{it})|\,dt$$

Step 2 : Differentiate m- more times

By direct computation, we have

$$D^m \frac{f'}{f} = -m! \sum_{k=1}^{N(R)} \frac{1}{(a_k - z)^{m+1}} + \underbrace{m! \sum_{k=1}^{N(R)} \frac{\bar{a}_k^{\,m+1}}{(R^2 - \bar{a}_k z)^{m+1}}}_{\text{term I}} + \underbrace{\text{Integral term}}_{\text{term II}}$$

where the integral term is

$$\frac{(m+1)!}{\pi} \int_0^{2\pi} \mathrm{Re} \, \frac{2Re^{it}}{(Re^{it} - z)^{m+2}} \log |f(Re^{it})| \, dt.$$

We show Term I & Term II converge to 0 as $R \to \infty$, yielding the Lemma. This will be achieved in the last two steps.

Step 3: Estimate term I.

$$\sum_{k=1}^{N(R)} \frac{\bar{a}_k^{\,m+1}}{(R^2 - \bar{a}_k z)^{m+1}} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

Let $R > 2|z|$. Since $\lambda < m+1$, we can pick ε with $\lambda + \varepsilon < m+1$

Note

$$|R^2 - \bar{a}_k z| \geq R^2 - |a_k| . |z| > R^2 - R \cdot \frac{R}{2} = \frac{R^2}{2}.$$

$$\Rightarrow \left| \frac{\bar{a}_k}{R^2 - \bar{a}_k z} \right|^{m+1} \leq \frac{R^{m+1}}{|(R^2/2)|^{m+1}} = \frac{2^{m+1}}{R^{m+1}}$$

$$\Rightarrow \left| \sum_{k=1}^{N(R)} \left(\frac{\bar{a}_k}{R^2 - \bar{a}_k z} \right)^{m+1} \right| \leq \sum_{k=1}^{N(R)} \left| \frac{\bar{a}_k}{R^2 - \bar{a}_k z} \right|^{m+1}$$

$$\leq N(R) \cdot \frac{2^{m+1}}{R^{m+1}} \leq (3R)^{\lambda+\varepsilon} \cdot \frac{2^{m+1}}{R^{m+1}} \longrightarrow 0$$

since $m+1 > \lambda + \varepsilon$.

Here, we used

$$N(R) < \log M(3R) < \log e^{(3R)^{\lambda+\varepsilon}} = (3R)^{\lambda+\varepsilon}.$$

Step 4 – Estimate the Integral (term II).

$$\int_0^{2\pi} \mathrm{Re}\, \frac{2Re^{it}}{(Re^{it}-2)^{m+2}} \log|f(Re^{it})|\, dt. \longrightarrow 0 \text{ as } R \to \infty$$

Claim

$$\int_0^{2\pi} \frac{2R\,e^{it}}{(Re^{it}-2)^{m+2}}\, dt \overset{w = Re^{it}}{=} \int_{|w|=R} \frac{2w}{(w-2)^{m+2}} \frac{dw}{iw} = 0$$

because the integrand admits an antiderivative.

Rewrite

$$\text{Term II} = \int_0^{2\pi} 2Re^{it} \frac{1}{(Re^{it}-2)^{m+2}} \log|f(Re^{it})|\, dt$$

Claim

$$= \int_0^{2\pi} 2Re^{it} \cdot \frac{1}{(Re^{it}-2)^{m+2}} \Big(\log|f(Re^{it})| - \log M(R)\Big)\, dt$$

$$\left|\text{Term } \underline{\overline{II}}\right| \leq \int_0^{2\pi} 2R \cdot \frac{1}{|Re^{it}-z|^{m+2}} \left(\log M(R) - \log |f(Re^{it})|\right) dt$$

$\Big\{$ using $|z| < R/2$

$$\leq \int_0^{2\pi} 2R \cdot \frac{1}{(R/2)^{m+2}} \left(\log M(R) - \log |f(Re^{it})|\right) dt$$

$$= \frac{2^{m+3}}{R^{m+1}} \cdot 2\pi \cdot \log M(R) - \frac{2^{m+3}}{R^{m+1}} \cdot \int_0^{2\pi} \log |f(Re^{it})|\, dt$$

$\Big\{$ Jensen's formula

$$\leq \frac{2^{m+3}}{R^{m+1}} \cdot 2\pi \cdot R^{\lambda+\varepsilon} - \frac{2^{m+3}}{R^{m+1}} \Bigg(\underbrace{\sum_{|a_k|<R} \log \left|\frac{R}{a_k}\right|}_{\text{positive contributions}} + \underbrace{\log |f(0)|}_{0} \Bigg)$$

$$\leq \frac{2^{m+3}}{R^{m+1}} \cdot 2\pi \cdot R^{\lambda+\varepsilon} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

This completes the proof.