Math 220C - Lecture 13

April 26, 2021

(1) Homework 6, available on Friday, due May 7

— last homework

(2) We drop the lowest homework

(3) Next 3 lectures – Little Picard.

In Lecture 11

Application A (Conway XI. 3. 6)

f entire & not constant & finite order

$\Rightarrow$ f omits at most one value.

Today — Picard's Theorems — Conway XII

Little Picard

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, non constant $\Rightarrow$ f omits at most one value.

For example, $f(z) = e^z$ only omits the value 0.

Little Picard is a generalization of

Liouville's Theorem $f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, non constant

$\Rightarrow$ f cannot be bounded.

Great Picard

$f: G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

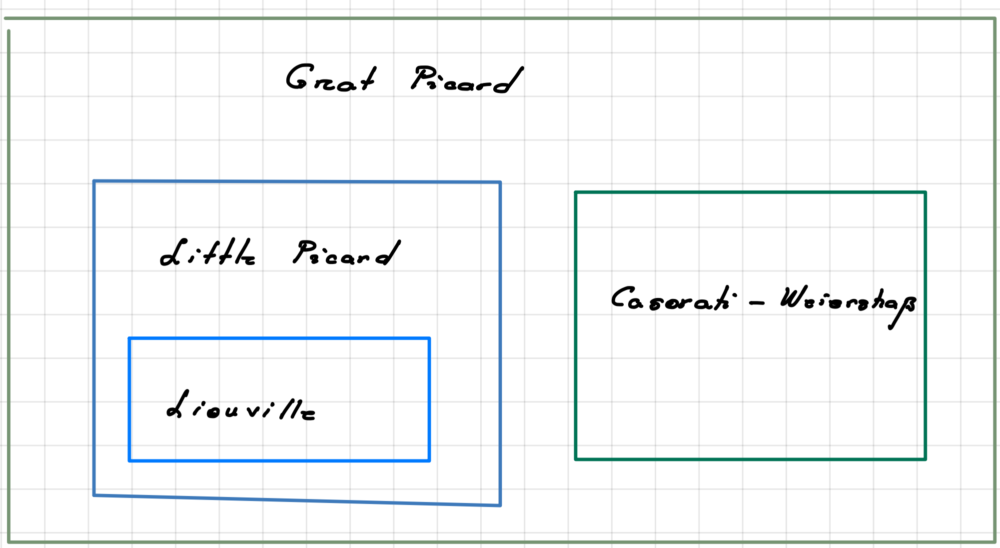
If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

Great Picard is a generalization of

Casorati–Weierstraß

$f: G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ has dense image in $\mathbb{C}$.



Great Picard > Little Picard   Conway XII. 4.4

Lemma

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, not polynomial.

$\Rightarrow$ f assumes all complex values ∞ - many times, with at most one exception.

Proof Let $g(z) = f\left(\frac{1}{z}\right) : \mathbb{C}^{\times} \longrightarrow \mathbb{C}$. Note that g has an essential singularity at 0. $\Leftrightarrow$ g does not have at worst a pole at 0 $\Leftrightarrow$ f does not have at worst a pole at ∞.

Recall from Math 220 A, Homework 5, Problem 6 that entire functions with poles at ∞ are polynomial, which is not the case for f.

Thus g has essential singularity at 0. Apply Great Picard to conclude.

We showed Great Picard $\Rightarrow$ Lemma $\Rightarrow$ Little Picard.

Examples

$\boxed{I}$ $e^f + e^g = 1$, f, g entire $\Rightarrow$ f, g constant.

Indeed, $h = e^f$ omits 0 & $h = 1 - e^g$ also omits 1.

Little Picard $\Rightarrow$ h constant $\Rightarrow$ f, g constant.

$\boxed{II}$ $f^n + g^n = 1$, $n \geq 3$, f, g entire $\Rightarrow$ f, g constant.

(HWK 6)

Emile Picard (1856 - 1941).

"Une fonction entiere, qui ne devient jamais ni a ni b est necessairment une constante" (Picard, 1879)

§2. Proof of Little Picard

Step A Landau's lemma — Conway XII. 2

Step B due to Bloch — Conway XII. 1

Assume $\exists f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant, omits 0 & 1.

Step A produces a function g entire and $\alpha > 0$ with

$\Delta \not\subset \operatorname{Im} g$ for all discs Δ of radius α

Step B For any g entire & not constant, $\operatorname{Im} g$ contains a disc of any radius, in particular of radius α.

Step A & Step B are incompatible, showing f does not exist $\Rightarrow$ Little Picard.

Landau's Lemma

Let $h : G \longrightarrow \mathbb{C}$ holomorphic, G simply connected

Assume h omits -1 & 1. Then $\exists\ F : G \longrightarrow \mathbb{C}$ holomorphic

such that $\boxed{h = \cos F.}$

Proof Note $1 - h^2$ is nowhere zero in G $\Rightarrow$ let g be a

square root of $1 - h^2$ $\Rightarrow$ $g^2 + h^2 = 1$ $\Rightarrow$ $(g + ih)(g - ih) = 1.$

Note $g + ih \neq 0$ in G $\Rightarrow$ $\exists$ logarithm for $g + ih$. Write

$$g + ih = e^{iF} \Rightarrow g - ih = \frac{1}{g + ih} = e^{-iF}$$

$$\Rightarrow g = \frac{1}{2}(e^{iF} + e^{-iF}) = \cos F.$$

Remark In our case f entire, omits 0 & 1 $\Rightarrow$

$\Rightarrow$ $2f-1$ omits -1 & 1 $\Rightarrow$ by Landau

$\Rightarrow$ $2f-1 = \cos \pi F$ & F entire.

Since $\cos \pi F = 2f-1 \neq \pm 1$ $\Rightarrow$ F omits all integers.

Thus F omits -1 & 1 and by Landau again

$\Rightarrow$ $F = \cos \pi g$ & $\cos \pi g$ is never an integer.

Conclusion

$$f = \frac{1}{2}(1 + \cos \pi F) = \frac{1}{2}(1 + \cos \pi \cos \pi g).$$

Define $A = \left\{ m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right) : n \in \mathbb{Z}_{>0},\ m \in \mathbb{Z} \right\}$

Let $\alpha^{\pm}_{mn} = m \pm \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right)$. Note

$$e^{i\pi\alpha^{+}_{mn}} = e^{i\pi m} \cdot e^{-\log\left(n+\sqrt{n^2-1}\right)} = (-1)^m \frac{1}{n+\sqrt{n^2-1}} = (-1)^m\left(n - \sqrt{n^2-1}\right).$$

$$e^{-i\pi\alpha^{+}_{mn}} = e^{-i\pi m} \cdot e^{\log\left(n+\sqrt{n^2-1}\right)} = (-1)^m\left(n + \sqrt{n^2-1}\right)$$

$$\Longrightarrow \cos \pi\alpha^{+}_{mn} = \frac{1}{2}\left(e^{i\pi\alpha^{+}_{mn}} + e^{-i\pi\alpha^{+}_{mn}}\right) = (-1)^m n \in \mathbb{Z}.$$

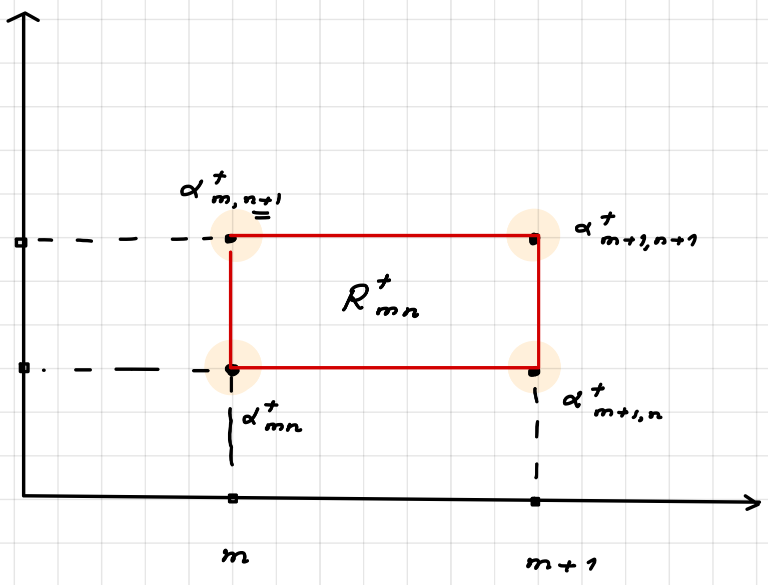
(The same argument works for α^{-}_{mn}.)

But $\cos \pi g$ cannot be an integer.

$\Downarrow$

Conclusion $\boxed{A \cap \operatorname{Img} = \emptyset.}$

Visualize A $\quad A = \left\{ m + \frac{i}{\pi} \log\left(n + \sqrt{n^2-1}\right) : n \in \mathbb{Z}_{>0},\ m \in \mathbb{Z} \right\}$



The set A gives the vertices of rectangles paving the plane. The upper half plane is paved by rectangles R^+_{mn}

- horizontal side $(m+1) - m = 1$

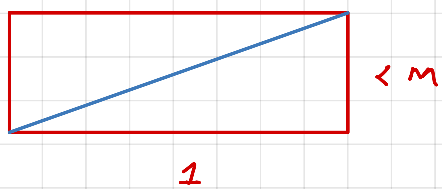
- vertical side $\frac{1}{\pi} \log\left(n+1 + \sqrt{(n+1)^2-1}\right) - \frac{1}{\pi} \log\left(n + \sqrt{n^2-1}\right) =$

$$= \frac{1}{\pi} \log \frac{n+1+\sqrt{(n+1)^2-1}}{n+\sqrt{n^2-1}} < M$$

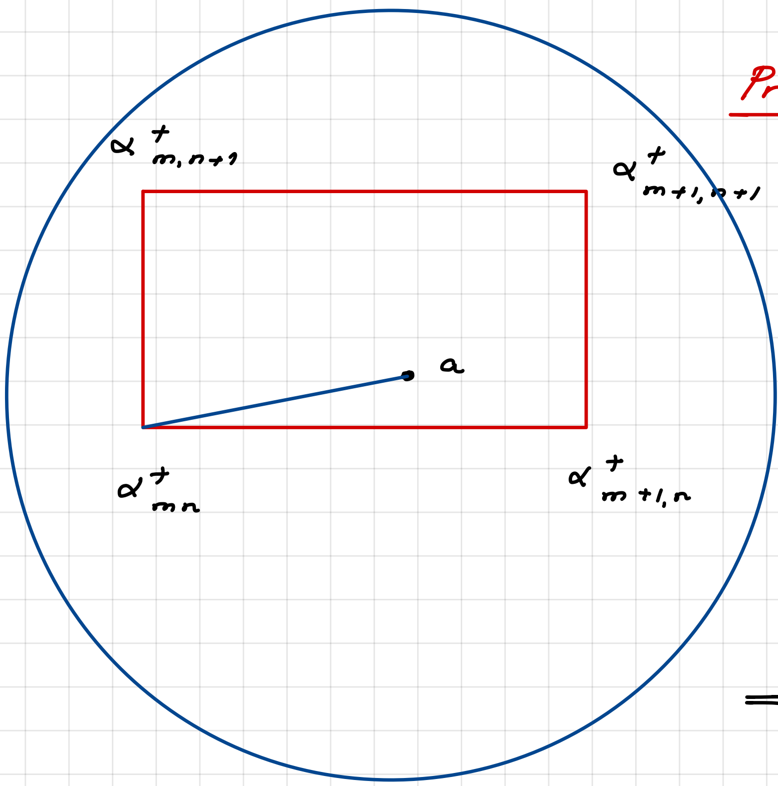
(make $n \to \infty$ to see boundedness).

The α^-_{mn}'s are used to pave the lower half plane.

The diameter of R^+_{mn} is $< \sqrt{1+M^2}$. Let $\alpha = \sqrt{2+M^2}$



Claim: If Δ is any disc of radius α then $\Delta \not\subset \mathrm{Img}$.



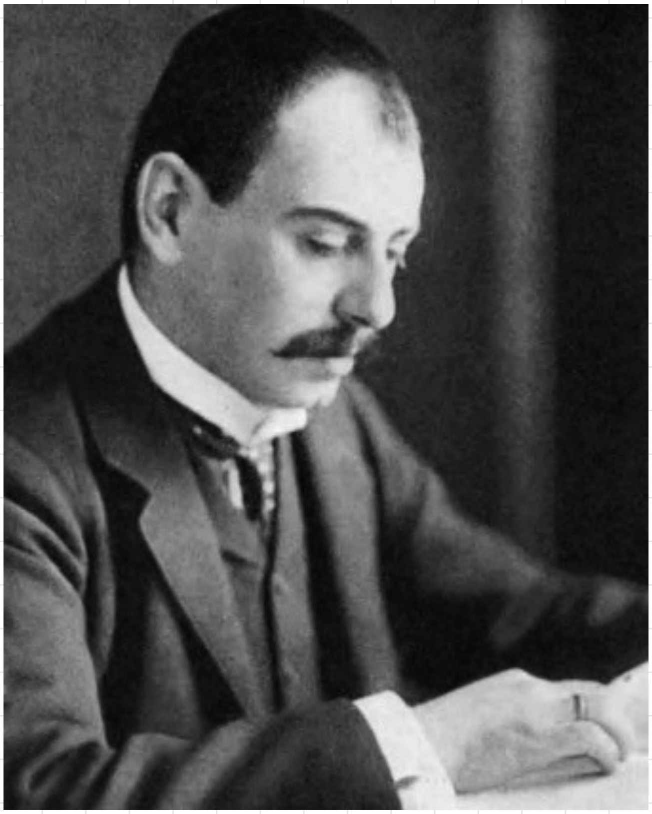
Proof: Let a be the center of Δ located say in the upper half plane. Then $a \in R^+_{mn}$.

$\Rightarrow |a - \alpha^+_{mn}| <$ diameter $(R^+_{mn}) < \alpha$

$\Rightarrow \alpha^+_{mn} \in \Delta$.

We have seen $\alpha^+_{mn} \notin \mathrm{Img}$. Thus $\Delta \not\subset \mathrm{Img}$.

This completes the proof of Step A. Step B will be discussed next.



Edmund Landau (1877 – 1938)

Big O - notation

Landau's Problems (ICM 1912)

- Goldbach's conjecture
- Twin prime conjecture
- Primes of the form n^2+1
- Primes between 2 consecutive perfect squares