# Math 220C - Lecture 14

April 28, 2021

# Last time — Strategy for proving Little Picard

Assume $\exists\, f: \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant, omits $0$ & $1$.

**Step A** produces a function $g$ entire, nonconstant and $\alpha > 0$ with $\Delta \not\subset \operatorname{Im} g$ for all discs $\Delta$ of radius $\alpha$

**Step B** For any $g$ entire & not constant, $\operatorname{Im} g$ contains a disc of any radius, in particular of radius $\alpha$.

Step A & Step B are incompatible, showing $f$ does not exist $\Rightarrow$ Little Picard.

## §1. Bloch's Theorem    Conway XII. 1.

**Notation**  $G$ open & bounded $\Rightarrow \overline{G}$ compact

$\mathcal{O}(\overline{G})$ = set of holomorphic functions in a neighborhood of $\overline{G}$

**Theorem** (version of Conway XII. 1.4).  $\Delta = \Delta(0,1)$.

Given $f \in \mathcal{O}(\overline{\Delta})$, $f'(0) = 1$, then $\operatorname{Im} f$ contains a disc a radius $\beta > 0$. In fact $\beta = \frac{3}{2} - \sqrt{2} \cong .085$ works

**Crucially** $\beta$ is a constant independent of the function $f$.

This is important for Little Picard.

**Remark** The value of $\beta$ in Conway is $\beta = \frac{1}{72} \cong .01$

This $\beta$ is smaller, however Conway proves a little more.

## Bloch $\Rightarrow$ ~~Step B~~ Conway XII. 2.

$g : \mathbb{C} \longrightarrow \mathbb{C}$ entire, not constant $\Rightarrow$ Im $g$ contains a disc of any radius.

**Proof** Fix a value $r$ for the radius.

$g$ not constant $\Rightarrow \exists\, a$ with $g'(a) \neq 0$. WLOG $a = 0$ else work with $g^{new}(z) = g(z+a)$.

Let $R > 0$. Define $h(z) = \dfrac{g(Rz)}{R\, g'(0)} \Rightarrow h'(0) = 1$ &

$h$ holomorphic in $\bar{\Delta} \Rightarrow$ Im $h$ contains a disc of radius $\beta$

$\Rightarrow$ Im $g$ contains a disc of radius $R\,|g'(0)|\,\beta > r$ if $R$ is chosen large.

**Remark** This completes the proof of Little Picard.

**Remark** The proof shows $g \in \mathcal{O}(\bar{\Delta}(a,R)) \Rightarrow$ Im $g$ contains a disc of radius $R|g'(a)|\beta$.

**Remark : Optimal value of $\beta$** Conway XII. 1. 9

Let $\mathcal{F} = \{ f \in \mathcal{O}(\bar{\Delta}),\ f'(0) = 1 \}$

$\boxed{i}$ $L_f = \{$ largest radius of a disc in $f(\Delta) \}$

$$L = \inf_{f \in \mathcal{F}} L_f = \text{Landau constant}$$

$\boxed{ii}$ $B_f = \{$ largest radius of a biholomorphic disc in $f(\Delta) \}$

$$B = \inf_{f \in \mathcal{F}} B_f = \text{Bloch constant}$$

$\boxed{iii}$ Current knowledge: $.5 < L < .544$

$$.433 < B < .472$$

Conjecturally $$B = \sqrt{\frac{\sqrt{3}-1}{2}}\ \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} \cong .471$$

We will show $L \geq \beta = \frac{3}{2} - \sqrt{2} \cong .08$.

2. Lemma. — *Die Funktion*

$$w = C\,\zeta\,\frac{\int\limits_0^1 t^{-\frac{1}{2}-\frac{1}{2k}}(1-t)^{-\frac{1}{6}+\frac{1}{2k}}(1-\zeta^3 t)^{-\frac{5}{6}+\frac{1}{2k}}\,dt}{\int\limits_0^1 t^{-\frac{1}{2}-\frac{1}{2k}}(1-t)^{-\frac{5}{6}+\frac{1}{2k}}(1-\zeta^3 t)^{-\frac{1}{6}+\frac{1}{2k}}\,dt}$$

*vermittelt die konforme Abbildung des Kreises* $|\zeta| < 1$ *auf ein gleichseitiges Kreisbogendreieck mit den Winkeln* $\pi/k$ $(k > 1)$. *Die Punkte* $1,\ \varepsilon,\ \varepsilon^2$ $\left(\varepsilon = \frac{-1+i\sqrt{3}}{2}\right)$ *entsprechen den Eckpunkten.*

Man findet die Abbildungsfunktion am einfachsten, wenn man von der Schwarzschen Beziehung

$$\{w, \zeta\} = \frac{9}{2}\left(1 - \frac{1}{k^2}\right)\frac{\zeta}{(\zeta^3-1)^2}$$

ausgeht und die assoziierte lineare Differentialgleichung

$$\frac{y''}{y} = -\frac{9}{4}\left(1 - \frac{1}{k^2}\right)\frac{\zeta}{(\zeta^3-1)^2}$$

zuerst durch die Substitution $y = (\zeta^3-1)^{\frac{1}{2}-\frac{1}{2k}}v,\ \zeta^3 = \xi$, dann durch $y = \zeta(\zeta^3-1)^{\frac{1}{2}-\frac{1}{2k}}u,\ \zeta^3 = \xi$ auf eine hypergeometrische reduziert.

Bestimmt man $C$ so, daß $w'(0) = 1$ wird, so ergibt sich

$$w(1) = \frac{\mathrm{B}\left(\frac{1}{2}-\frac{1}{2k},\ \frac{1}{6}+\frac{1}{2k}\right)}{\mathrm{B}\left(\frac{1}{2}-\frac{1}{2k},\ \frac{5}{6}+\frac{1}{2k}\right)} = \frac{\Gamma\left(\frac{1}{6}+\frac{1}{2k}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}+\frac{1}{2k}\right)\Gamma\left(\frac{2}{3}\right)}.$$

Die uns interessierenden Fälle sind $k = 3$ und $k = 6$. Man findet durch Einsetzung

$$\lambda = \frac{w_2(1)}{w_1(1)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma(1)\,\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}. \tag{3}$$

Aus (1), (2) und (3) erhält man jetzt endlich

$$\mathfrak{B}' = \sqrt{\frac{\sqrt{3}-1}{2}}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} = \sqrt{\pi}\cdot 2^{1/4}\cdot\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(\frac{\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{12}\right)}\right)^{1/2} = 0{,}4719\ldots$$

und es ist somit bewiesen, daß

$$\mathfrak{B} \leqq \mathfrak{B}' < 0{,}472.$$

Früher bekannt war die Landausche Abschätzung[1])

$$0{,}396 < \mathfrak{B} < 0{,}555.$$

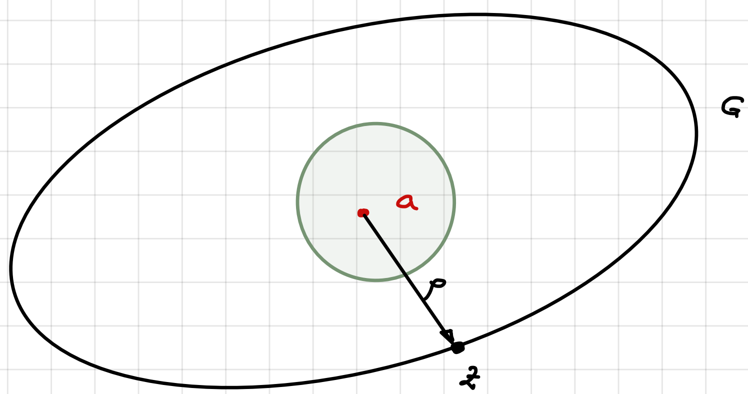
[1]) Landau, Über die Blochsche Konstante und zwei verwandte Weltkonstanten. Mathem. Zeitschr. 30 (1929), 608—634, insbesondere S. 614.

Ahlfors - Grunski, Math. Zeitscrift, 42 (1937)

## §2. Proof of Bloch's theorem

**Question** How can we construct a disc in $\operatorname{Im} f$?

Assume $G$ bounded, $a \in G$. Let $\rho = \min_{z \in \partial G} |f(z) - f(a)|$




**Lemma A** Let $f \in \mathcal{O}(\bar{G})$, $a \in G$, $\rho = \min_{z \in \partial G} |f(z) - f(a)|$.

Then $\operatorname{Im} f$ contains $\Delta(f(a), \rho)$.

**Remark** This can be viewed as a more precise Open Mapping theorem.

**Proof** Let $H = f(G) \subseteq f(\bar{G}) =$ compact since $\bar{G}$ is compact. Then $H$ is bounded $\Rightarrow \partial H$ compact. Let

$$R = d(f(a), \partial H) = \min_{h \in \partial H} |h - f(a)|.$$

$\Rightarrow \Delta(f(a), R) \subseteq H \subseteq \operatorname{Im} f$. We show

$$R \geq \rho$$

$\Rightarrow \Delta(f(a), \rho) \subseteq \operatorname{Im} f$ proving Lemma A.

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To prove $R \geq \rho$, let $w \in \partial H$ achieve the minimum $R$.

**Claim** $w = f(z), \; z \in \partial G$.

Then $R = |w - f(a)| = |f(z) - f(a)| \geq \rho$ by definition of $\rho$ as a minimum.

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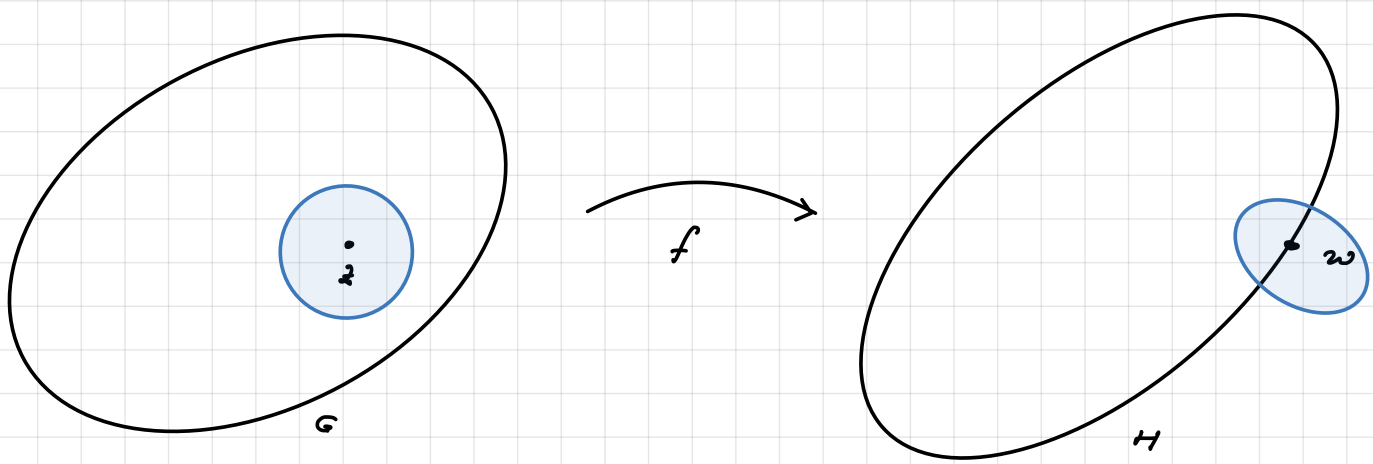
**Proof of the Claim** Since $w \in \partial H \Rightarrow \exists\, h_n \in H,\ h_n \to w$.

Write $h_n = f(g_n),\ g_n \in G \subseteq \bar{G}$. Since $\bar{G}$ compact $\Rightarrow$ passing to a subsequence we may assume that $g_n \longrightarrow z \in \overline{G}$. $\Rightarrow$ $f(g_n) \longrightarrow f(z)$. Since

$f(g_n) = h_n \longrightarrow w$, we conclude

$$w = f(z)., \quad z \in \overline{G}$$

If $z \in G$, we contradict Open Mapping theorem.




( Pick a disc near $z$. its image will be open so it will contain a disc near $w$ but $w \in \partial H$ contradiction).

Thus $z \in \partial G$ proving the Claim & Lemma A.

## Strategy for Bloch

Apply Lemma A & show $|f(z) - f(a)| \geq \beta$ for suitable a.

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**Question**: Why is the proof difficult?

**Answer**: We don't know a. In other words, we don't know where the center of the disc in Bloch should be.

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## More detailed strategy

[I] prove Bloch under Assumption (*)

[II] remove Assumption (*)

In Step [I] we have control of the center & the radius equals $2\beta$ (better than Bloch claims).

In Step [II] we lose control of center, radius halves, but we have no assumptions.

## Assumption (*)

$$f \in \mathcal{O}(\bar{\Delta}), \quad |f'(z)| \le 2|f'(0)| \quad \forall\ |z| \le 1$$

## Lemma B (Bloch assuming (*))

If $f$ satisfies Assumption (*) $\Rightarrow$ $\mathrm{Im}\, f$ contains a disc with center $f(0)$ & radius $2\beta|f'(0)|$

**Remark** If $f'(0) = 1$, this implies under Assumption (*)

## Lemma C (Bloch without (*)).

For all $f \in \mathcal{O}(\bar{\Delta})$, even in the absence of Assumption (*), $\mathrm{Im}\, f$ contains a disc of radius $\beta\,|f'(0)|$.

Note Lemma C $\Rightarrow$ Bloch.

Next time we show Lemma B $\Rightarrow$ Lemma C.