Math 220C - Lecture 15

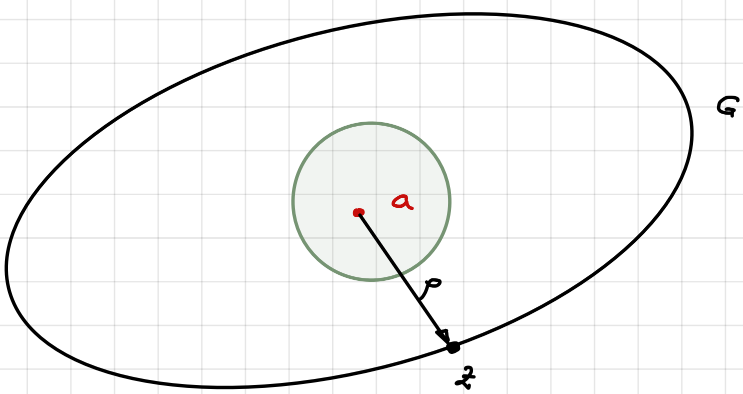
April 30, 2021

. Last time

Theorem (version of Conway XII. 1.4). $\Delta = \Delta(0,1)$.

Given $f \in \mathcal{O}(\bar{\Delta})$, $f'(0) = 1$, then $\operatorname{Im} f$ contains a disc a radius $\beta > 0$. In fact $\beta = \frac{3}{2} - \sqrt{2} \cong .085$ works

Main Tool — Sharper Open Mapping Theorem



Lemma A Let $f \in \mathcal{O}(\bar{G})$, $a \in G$, $\rho = \min_{z \in \partial G} |f(z) - f(a)|$.

Then $\operatorname{Im} f$ contains $\Delta(f(a), \rho)$.

Lemma B (Stronger form of Bloch but with assumptions)

If $f \in \mathcal{O}(\bar{\Delta}(a,R))$ and $|f'(z)| \leq 2|f'(a)|$ in $\bar{\Delta}(a,R)$,

then $\operatorname{Im} f$ contains a disc of radius $2\beta\, |f'(a)|\, R$.

Remark $R=1$, $a=0$, $f'(0)=1$ is Bloch under the assumption $|f'(z)| \leq 2$. We get a disc of radius 2β!

Remark We state this for all centers a since we don't know where our center will end up.

Proof WLOG $R=1$ & $a=0$, else rescale & translate.

WLOG $f(0)=0$ else work with $f-f(0)$.

Hypothesis $|f'(z)| \leq 2|f'(0)|$ for $|z| \leq 1$.

Goal Disc of radius $2\beta\, |f'(0)|$.

Plan Estimate $\boxed{\rho = \min |f(z) - f(0)|}$ when $|z| = r$.

& Apply **Lemma A** $\Rightarrow$ $\boxed{\operatorname{Im} f \supseteq \Delta(f(0), \rho).}$

Estimate for ρ

Let $F(z) = f(z) - z f'(0) =$

$$= \int_0^z \left(f'(w) - f'(0)\right) dw \qquad w = zt,\ 0 \le t \le 1$$

$$= \int_0^1 \left(f'(tz) - f'(0)\right) z\, dt$$

Apply **Cauchy Integral Formula**

$$f'(tz) - f'(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f'(\zeta)}{\zeta - tz}\, d\zeta - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f'(\zeta)}{\zeta}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta|=1} f'(\zeta) \left(\frac{1}{\zeta - tz} - \frac{1}{\zeta} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta|=1} f'(\zeta) \cdot \frac{tz}{\zeta(\zeta - tz)}\, d\zeta$$

Substituting,

$$F(z) = \frac{1}{2\pi i} \int_0^1 \int_{|\zeta|=1} f'(\zeta) \cdot \frac{tz^2}{\zeta(\zeta - tz)} \cdot d\zeta\, dt$$

Take absolute values. Note

Hypothesis

$$\left| f'(\zeta) \cdot \frac{tz}{\zeta(\zeta - tz)} \right| \leq 2|f'(0)| \cdot \frac{t|z|}{1 \cdot (1-|z|)}$$

since $|\zeta - tz| \geq |\zeta| - |tz| = 1 - t|z| \geq 1 - |z|$.

Therefore

$$|F(z)| \leq \frac{1}{2\pi} \cdot \int_0^1 2|f'(0)| \cdot \frac{t|z|^2}{1-|z|} \cdot dt \cdot \underbrace{\text{length}(|\zeta| = 1)}_{2\pi}$$

$$= 2|f'(0)| \cdot \frac{r^2}{1-r} \cdot \underbrace{\int_0^1 t\, dt}_{1/2}$$

$$= |f'(0)| \cdot \frac{r^2}{1-r} \quad \text{for } |z| = r. \qquad (1)$$

On the other hand, by triangle inequality

$$|F(z)| = |z f'(0) - f(z)| \geq |z f'(0)| - |f(z)|$$

$$= r|f'(0)| - |f(z)| \qquad (2)$$

Using (1) & (2) we find

$$r|f'(0)| - |f(z)| \leq |f'(0)| \cdot \frac{r^2}{1-r}$$

$\Rightarrow \ |f(z)| \geq |f'(0)| \cdot \left(r - \frac{r^2}{1-r}\right)$. for $|z| = r$.

We haven't specified r yet. In any case, from Lemma A applied to $f|_{\bar{\Delta}(0,r)}$, the image of f contains a disc of radius

$$|f'(0)| \left(r - \frac{r^2}{1-r}\right).$$

To get the best radius, we maximize

$$r - \frac{r^2}{1-r}.$$

The critical point is $r_0 = 1 - \frac{1}{\sqrt{2}}$, maximum value equals 2β.

We obtain a disc of radius $2\beta \, |f'(0)|$.

Lemma B $\Rightarrow$ Bloch We show

For all $f \in \mathcal{O}(\overline{\Delta})$, $\operatorname{Im} f$ contains a disc of radius $\beta\,|f'(0)|$.

When $f'(0) = 1$, this is exactly Bloch's theorem.

Proof Let $h(z) = |f'(z)|\,(1-|z|)$ continuous in $\overline{\Delta}$.

Let M be the maximum of h achieved at p.

Let $1 - |p| = 2t$ $\Rightarrow$

$$M = h(p) = |f'(p)|\,(1-|p|) = 2t\,|f'(p)|.$$

If $z \in \overline{\Delta}(p,t)$ then $|z-p| \le t$, $|p| = 1-2t$ $\Rightarrow$ $|z| \le |z-p| + |p| \le 1-t$

$\Rightarrow$ $1 - |z| \ge t$. Therefore since p is a maximum,

$$\underbrace{(1-|z|)}_{\ge t}\,|f'(z)| \le \underbrace{(1-|p|)}_{2t}\,|f'(p)| \Rightarrow |f'(z)| \le 2\,|f'(p)|.$$

in $\overline{\Delta}(p,t)$.

Apply Lemma B to $f/_{\bar{\Delta}(p,t)}$ $\Rightarrow$ the image of f

contains a disc of center $f(p)$ and radius

$$2\beta\, |f'(p)|\, t = \beta M.$$

Note $M = \max_{|z|\le 1} h \ge h(0) = |f'(0)|$ $\Rightarrow$ the disc we constructed

has radius $\beta M \ge \beta\, |f'(0)|$.

This completes the proof of Bloch & Little Picard along with it.

Remark* (will not use)

The reason for our choice of h is not transparent

The choice of center is also mysterious. We motivate

these choices below.

Question What is the most natural h ?

Answer We seek to achieve $|f'(z)| \leq 2|f'(0)|$. & use Lemma B.

We have a better chance if we maximize $|f'(0)|$

What happens if we replace f by $f \circ \varphi_{-\alpha}$ where $\varphi_{-\alpha} \in \mathrm{Aut}\,\Delta$. ?

Note

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$$|(f \circ \varphi_{-\alpha})'(0)| = |f'(\varphi_{-\alpha}(0)) \cdot \varphi_{-\alpha}'(0)| = |f'(\alpha)|\,(1-|\alpha|^2).$$

Thus to get a larger derivative, we are led to maximizing

$$\tilde{h}(z) = |f'(z)|\,(1-|z|^2)$$

which is similar to what we used. This also suggests the new center is $\varphi_{-\alpha}(0) = \alpha$, which is also what we used.

Exercise Run the above argument using $\tilde{h}$ instead of h.

Remark* (will not use) Original proof of Little Picard.

Sketch:

Construct $\lambda : \mathfrak{H}^+ \longrightarrow \mathbb{C} \setminus \{0,1\}$ universal cover

In the diagram

$$\begin{tikzcd} & & \mathfrak{H}^+ \arrow[r, "\text{Cayley}", "\sim"', "C"'] \arrow[d, "\lambda"] & \Delta \\ f : \mathbb{C} \arrow[rr] \arrow[urr, dashed, "\tilde{f}"] & & \mathbb{C} \setminus \{0,1\} & \end{tikzcd}$$

show we can lift f to a holomorphic function $\tilde{f}$.

Then $C \circ \tilde{f} : \mathbb{C} \longrightarrow \Delta$ is entire & bounded $\Rightarrow C \circ \tilde{f}$ is constant $\Rightarrow \tilde{f}$ constant $\Rightarrow f = \lambda \circ \tilde{f}$ is constant QED.

The crux of the matter is the construction of λ

(1) λ holomorphic

(2) λ Γ-invariant, Γ = Deck transformations.

λ is a modular function for the group $\Gamma(2)$.

(1) End of material for Qualifying Exam.

(2) Qualifying Exam – May 18, 5 – 8 PM.

(3) closed book, notes, internet, via Gradescope

(4) covers Math 220 AB & Math 220C up to and including Lecture 15

(5) Past Qualifying Exams are linked on website

(6) Review – closer to the date (May 14 ? May 17 ?)

(7) Last homework – Homework 6.

What is next?

(1) Great Picard — Conway XII.3, XII.4.

(2) An introduction to Riemann Surfaces.