Math 220C - Lecture 16

May 3, 2021

Lecture 14 – Proof of Little Picard (summary)

Step A $f : G \longrightarrow \mathbb{C} \setminus \{0,1\}$, G simply connected

(1) Write $f = \frac{1}{2}(1 + \cos \pi \cos \pi g)$.

(2) $\operatorname{Im} g$ contains no disc of radius α.

(3) $h(z) = \dfrac{g(Rz)}{R\, g'(0)}$, $h \in \mathcal{O}(\bar{\Delta})$, $h'(0) = 1$.

If $R \gg 0$, we showed h contradicts Bloch.

Step B (Bloch – Lecture 15)

$h \in \mathcal{O}(\bar{\Delta})$, $h'(0) = 1 \Rightarrow \operatorname{Im} h$ contains a disc of radius β

Road map to Great Picard

$f: G \setminus \{a\} \to \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

$$\begin{tikzcd}
 & \text{Bloch \& Landau} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Schottky (today)} \arrow[d, Rightarrow] & & \text{Little Picard} \\
\text{Strong Montel (next time)} \arrow[d, Rightarrow] & & \\
\text{Great Picard (next time)} & &
\end{tikzcd}$$

The broad goal is to study the family

$$\mathcal{F} = \{ f : G \longrightarrow \mathbb{C} \setminus \{0,1\} \text{ holomorphic} \}.$$

When $G = \mathbb{C}$, $\mathcal{F}$ consists of constant functions.

by Little Picard

When $G = \Delta^*(a,r)$ this is relevant for Great Picard

Question Is $\mathcal{F}$ normal?

Remark To answer this question we need uniform bounds on $|f(z)|$ in small discs.

Schottky's theorem

$\exists$ function $c(a,b)$ for $0 < a < \infty$, $0 < b < 1$, increasing in each variable so that

$\forall f \in \mathcal{O}(\bar{\Delta})$ omitting 0 & 1, $|f(0)| = a$, then

$$|f(z)| \leq c(a,b) \quad \text{if } |z| \leq b$$

Remark The theorem controls the growth of $f \in \mathcal{F}$ in a universal fashion provided $|f(0)| = $ fixed.

Remark We will show that

$$c(a,b) = \frac{1}{2} + \frac{1}{2} \exp \pi \exp \pi \left(3 + 2a + \frac{\alpha}{\beta} \cdot \frac{b}{1-b} \right).$$

Key Claim

For each $z \in \mathbb{C}$, the equation $\cos \pi a = z$ admits a solution

$$|a| \leq 1 + |z|.$$

Proof It is easy to check that $\cos \pi a = z$ admits a solution a. by converting into a quadratic equation in $w = e^{\pi i a}$ using $\cos \pi a = \frac{w + w^{-1}}{2} = z$

Note that if a is a solution, $a + 2$ is also a solution.

Thus we may assume $\mathrm{Re}\, a \in [-1, 1] \Rightarrow |\mathrm{Re}\, a| \leq 1$.

Then

$$|a| \leq |\mathrm{Re}\, a| + |\mathrm{Im}\, a| \overset{(*)}{\leq} 1 + |\cos \pi a| = 1 + |z|.$$

Inequality (*) $\quad |\mathrm{Im}\, a| \leq |\cos \pi a|$

Proof $\quad a = x + iy$

$$\cos \pi a = \frac{e^{\pi a i} + e^{-\pi a i}}{2} =$$

$$= \frac{e^{\pi x i} e^{-\pi y} + e^{-\pi x i} \cdot e^{\pi y}}{2}$$

$$= \cos \pi x \left(\frac{e^{\pi y} + e^{-\pi y}}{2} \right) + i \sin \pi x \left(\frac{e^{-\pi y} - e^{\pi y}}{2} \right)$$

$$\Rightarrow |\cos \pi a|^2 = \cos^2 \pi x \left(\frac{e^{\pi y} + e^{-\pi y}}{2} \right)^2 + \sin^2 \pi x \left(\frac{e^{-\pi y} - e^{\pi y}}{2} \right)^2$$

$$= \left(\frac{e^{-\pi y} - e^{\pi y}}{2} \right)^2 + \cos^2 \pi x$$

$$= \sinh^2 \pi y + \cos^2 \pi x$$

$$\geq \sinh^2 \pi y \geq (\pi y)^2 > y^2 . = |\operatorname{Im} a|^2.$$

This completes the proof.

Proof of Schottky's theorem

Step 1 Revisit Landau's Lemma

Let $f \in \mathcal{O}(\bar{\Delta})$ omitting 0 & 1 $\Rightarrow$ $2f-1$ omits -1 & 1.

By Landau

$$2f-1 = \cos \pi F \Rightarrow 2f(0)-1 = \cos \pi F(0).$$

By Key Claim, we may assume

$$|F(0)| \leq 1 + |2f(0)-1|$$

By Lecture 13, F omits ± 1. We write

$$F = \cos \pi g \Rightarrow F(0) = \cos \pi g(0).$$

By Key Claim, we may assume

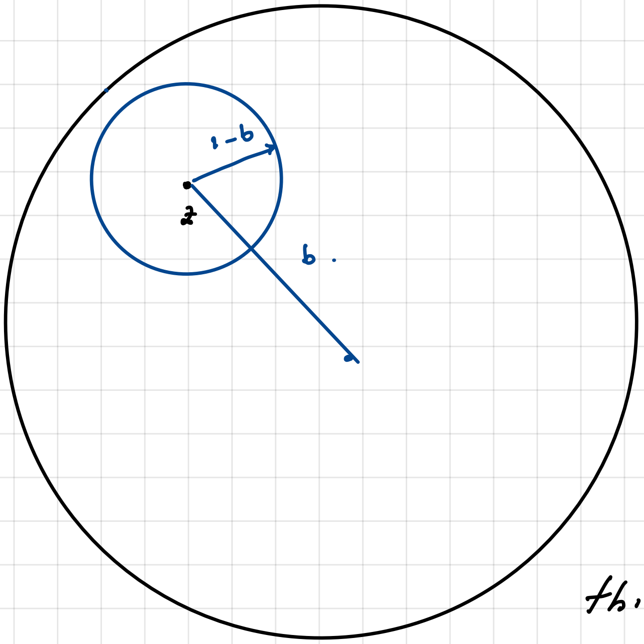
$$|g(0)| \leq 1 + |F(0)|. \leq 1 + 1 + |2f(0)-1| \leq 3 + 2|f(0)|.$$

Conclusion $f = \frac{1}{2}(1 + \cos \pi \cos \pi g)$ &

$$|g(0)| \leq 3 + 2a \quad \text{if } |f(0)| = a.$$

Step 2 Bounding g'

Let $|z| \leq b$. $\Rightarrow \bar{\Delta}(z, 1-b) \subseteq \bar{\Delta}$.



Define

$$h(w) = \frac{g(z + (1-b)w)}{(1-b)\, g'(z)}$$

(recenter & rescale). Compare this to item (3) Step A in Little Picard.

$\Rightarrow h \in \mathcal{O}(\bar{\Delta})$, $h'(0) = 1$. By Bloch

$\Rightarrow \operatorname{Im} h$ contains a disc of radius β

$\Rightarrow \operatorname{Im} g$ contains a disc of radius $\beta(1-b)|g'(z)|$

We showed in Lecture 13, $\operatorname{Im} g$ contains no disc of radius α

$$\Rightarrow \alpha \geq \beta(1-b)|g'(z)|$$

$$\Rightarrow |g'(z)| \leq \frac{\alpha}{\beta} \cdot \frac{1}{1-b}. \quad \forall\, |z| \leq b.$$

Step 3 Bounding g and f

We have shown $\boxed{|g'(z)| \leq \frac{\alpha}{\beta} \cdot \frac{1}{1-b}}$ if $|z| \leq b$.

Note

$$|g(z)| = \left| g(0) + \int_0^z g'(w)\, dw \right|$$

$$\leq |g(0)| + \left| \int_0^z g'(w)\, dw \right|$$

step 1, step 2

$$\leq (3+2a) + \left(\frac{\alpha}{\beta} \cdot \frac{1}{1-b} \right) |z|$$

$$\leq (3+2a) + \frac{\alpha}{\beta} \cdot \frac{b}{1-b} \quad \forall\ |z| = b,\ |f(0)| = a.$$

To bound f, we need

Claim $|\cos w| \leq \exp |w|$

Proof $|\cos w| = \left| \frac{e^{iw} + e^{-iw}}{2} \right| \leq \frac{|e^{iw}| + |e^{-iw}|}{2}$

$$= \frac{e^{Re(iw)} + e^{Re(-iw)}}{2}$$

$$\leq \frac{e^{|iw|} + e^{|-iw|}}{2} = e^{|w|}.$$

Now we can finish the argument

$$|f(z)| = \left|\frac{1}{2} + \frac{1}{2}\cos\pi\cos\pi g(z)\right|$$

$$\leq \frac{1}{2} + \frac{1}{2}\left|\cos\pi\cos\pi g(z)\right|$$

Claim

$$\leq \frac{1}{2} + \frac{1}{2}\exp\pi\left|\cos\pi g(z)\right|$$

Claim

$$\leq \frac{1}{2} + \frac{1}{2}\exp\pi\exp\pi|g(z)|$$

$$\leq \frac{1}{2} + \frac{1}{2}\exp\pi\exp\pi\left(3 + 2a + \frac{\alpha}{\beta}\cdot\frac{b}{1-b}\right)$$

$$= c(a,b) \quad \text{if} \quad |f(0)| = a, \quad |z| = b.$$

Friedrich Schottky

(1851 - 1935)

Academic advisors

Karl Weierstrass

Worked on elliptic, abelian,
and theta functions.

Schottky problem:

Characterization of Jacobian varieties amongst abelian varieties.

The author is of a clumsy appearance, unprepossessing, a dreamer, but if I'm not completely wrong, he possesses an important mathematical talent. [...] As rector I had to cancel his name from the register because neither had he attended lectures nor were his whereabouts in Berlin known. (Weierstrass.)

https://mathshistory.st-andrews.ac.uk/Biographies/Schottky/