Math 220C - Lecture 17

May 5, 2021

Last time — Schottky's Theorem

$\exists$ function $c(a,b)$ for $0 < a < \infty$, $0 < b < 1$, increasing in each variable so that

$\forall f \in \mathcal{O}(\overline{\Delta}(0,R))$ omitting 0 & 1, $|f(0)| = a$, then

$$|f(z)| \leq c(a,b) \text{ if } |z| \leq bR.$$

Remark Last time, $R = 1$. The above statement follows by rescaling

$$f^{new}(z) = f(Rz).$$

§1. Strong Montel – Conway XII.4

$$\mathcal{F} = \{ f \text{ holomorphic in } G, \text{ omitting } 0 \,\&\, 1 \}.$$

Question Is $\mathcal{F}$ normal?

Answer No! If $G = \mathbb{C}$, $\mathcal{F}$ consists of constants by Little Picard.

If $f_n = c_n = \text{constant}$, it may happen that $c_n \to \infty$.

We modify the definition of a normal family:

Definition A family $\mathcal{F}$ is normal in the extended sense if every sequence in $\mathcal{F}$ admits a subsequence converging locally uniformly to a function or converging locally uniformly to ∞.

Strong Montel Theorem

The family

$$\mathcal{F} = \{ f \text{ holomorphic in } G, \text{ omitting } 0 \,\&\, 1 \}.$$

is normal in the extended sense.

Remark This is clear for $G = \mathbb{C}$ by Little Picard.

Remark In Math 220B, we showed

$\mathcal{H}$ normal $\iff$ $\mathcal{H}$ locally bounded.

Proof Fix $z_0 \in G$. Define

$$\mathcal{F}^+ = \{ f \in \mathcal{F}, \ |f(z_0)| \le 1 \}$$

$$\mathcal{F}^- = \{ f \in \mathcal{F}, \ |f(z_0)| \ge 1 \}$$

$$\implies \mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-.$$

Claim $\mathcal{F}^+$ locally bounded.

Thus $\mathcal{F}^+$ is normal.

Let $f_n \in \mathcal{F}$ be a sequence. Either

[i] ∞ - many terms of f_n are in $\mathcal{F}^+$

[ii] ∞ - many terms of f_n are in $\mathcal{F}^-$

In case [i], we may assume $f_n \in \mathcal{F}^+$ after relabelling.

$\mathcal{F}^+$ normal shows $\{f_n\}$ has a convergent subsequence

as needed.

In case [ii], we may assume $f_n \in \mathcal{F}^-$ after relabelling.

$\Rightarrow$ $1/f_n \in \mathcal{F}^+$. Passing to a subsequence, we may assume

$1/f_n \overset{l.u.}{\rightrightarrows} \varphi$ since $\mathcal{F}^+$ is normal. Since $1/f_n$ is zero-free

$\Rightarrow$ φ is zero-free or $\varphi \equiv 0$ by Hurwitz's theorem.

If φ zero-free, $f_n \overset{l.u.}{\rightrightarrows} 1/\varphi$. If $\varphi \equiv 0$, $1/f_n \overset{l.u.}{\rightrightarrows} 0$ so

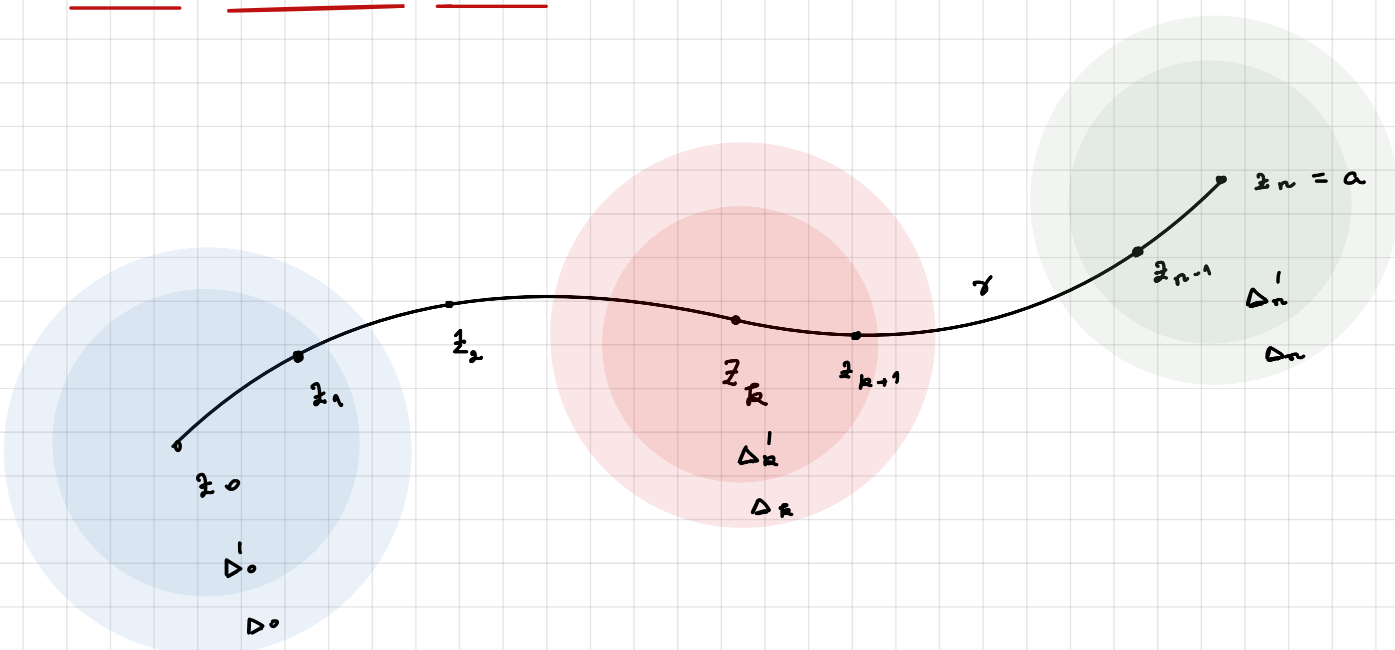
$f_n \overset{l.u.}{\rightrightarrows} \infty$. This is what we needed.

Math 220 A - Lecture 24

Hurwitz $f_n \overset{l.u.}{\rightrightarrows} f$, f_n holomorphic in $\mathcal{U}$,

If f_n is zero free $\forall n \Rightarrow f$ zero-free or $f \equiv 0$.

Proof of the claim



We know $|f(z_0)| \le 1$ for $f \in \mathcal{F}^+$. We seek to bound f near any $a \in G$.

Let γ be a path joining z_0 to a. Cover γ by

- discs $\Delta_0', \Delta_1', \dots, \Delta_n'$ of centers $z_0, \dots, z_n = a$
- such that $\bar{\Delta}_k' \subseteq \Delta_k \subseteq G \quad \forall k$.
- $z_{k+1} \in \Delta_k' \quad \forall k$

This can be done via a compactness argument.

Apply Schottky's Theorem in $\Delta_0' = \Delta_0'(z_0, R_0)$. Note $f \in \mathcal{O}(\bar{\Delta}_0')$. Then $z_1 \in \Delta_0'$ so

$$|f(z_1)| \leq C\left(|f(z_0)|, \frac{|z_1 - z_0|}{R_0}\right) \leq C\left(1, \frac{|z_1 - z_0|}{R_0}\right) := C_1$$

Apply Schottky's Theorem in $\Delta_1' = \Delta_1'(z_1, R_1)$.

$$|f(z_2)| \leq C\left(|f(z_1)|, \frac{|z_2 - z_1|}{R_1}\right) \leq C\left(C_1, \frac{|z_2 - z_1|}{R_1}\right) := C_2$$

Continue in this fashion. We obtain

$$|f(z_n)| \leq C_n. \implies |f(a)| \leq C_n \text{ since } z_n = a.$$

Apply Schottky one more time in $\overline{\Delta}(a, \frac{R_n}{2})$.

$$|f(z)| \leq C\left(|f(z_n)|, \frac{|z_n - z|}{R_n}\right) \leq C\left(c_n, \frac{1}{2}\right) := M.$$

Thus $|f(z)| \leq M$ $\forall f \in \mathcal{F}$, in the disc $\overline{\Delta}(a, R/2)$.

This proves the claim & Strong Montel.

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§2. Great Picard

$f : G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic, with essential singularity at a.

If $\Delta^*(a,r) \subseteq G \setminus \{a\}$, then $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.

Remark The proof is very similar to Math 220B, Midterm 2

Problem 2. [*10 points.*]

Let $f : \Delta(0,1) \setminus \{0\} \to \mathbb{C}$ be a holomorphic function on the punctured unit disc. Let

$$f_n : \Delta(0,1) \setminus \{0\} \to \mathbb{C}, \quad f_n(z) = f\left(\frac{z}{n}\right).$$

Show that the family $\mathcal{F} = \{f_n : n \geq 1\}$ is normal iff f has a removable singularity at the origin.

Proof We show $f\big|_{\Delta^*(a,r)}$ omits at most one value.

WLOG $a = 0$. Write $\Delta^* = \Delta(0,r) \setminus \{0\}$.

Assume $f\big|_{\Delta^*}$ omits two values, say 0 & 1.

Let $\mathcal{F} = \{ f : \Delta^* \longrightarrow \mathbb{C},\ f \text{ omits } 0 \ \& \ 1 \}$. $\Rightarrow$ $\mathcal{F}$ normal in the extended sense. Let

$$f_n(z) = f\left(\frac{z}{n}\right) \Rightarrow f_n \in \mathcal{F}.$$

Thus $\{f_n\}$ is normal in the extended sense being a subfamily of $\mathcal{F}$.

Thus $f_{n_k} \overset{\text{l.u.}}{\rightrightarrows} \varphi$ or $f_{n_k} \overset{\text{l.u.}}{\rightrightarrows} \infty$.

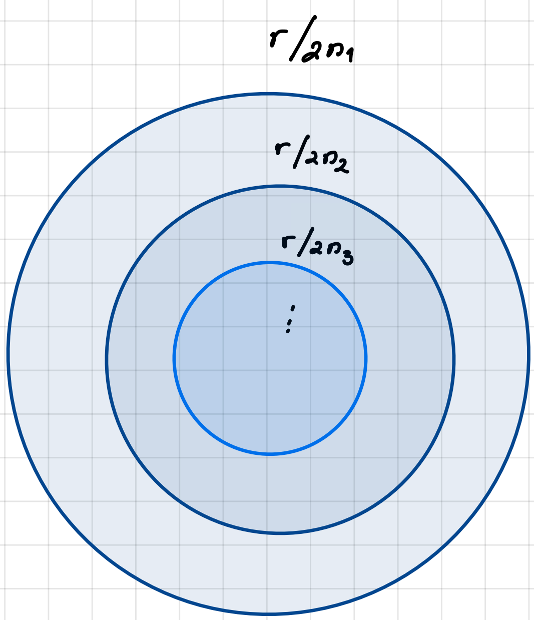
In the first case we show f has a removable singularity.

This follows from the midterm problem whose argument we recall. Since φ cont. for $|z| = \frac{r}{2}$ $\Rightarrow$ $|\varphi(z)| \leq M$ for $|z| = \frac{r}{2}$

Since $f_{n_k} \rightrightarrows \varphi$ for $|z| = \frac{r}{2}$ $\Rightarrow$ $|f_{n_k}(z) - \varphi(z)| \leq M$ for $|z| = \frac{r}{2}$

$\Rightarrow$ $|f_{n_k}(z)| \leq |f_{n_k}(z) - \varphi(z)| + |\varphi(z)| \leq M + M = 2M$ $\forall$ $|z| = \frac{r}{2}$

$\Rightarrow$ $|f(w)| \leq 2M$ $\forall$ $|w| = \frac{r}{2n_k}$.



By maximum principle,

$|f(w)| \leq 2M$ for $\frac{r}{2n_{k+1}} \leq |w| \leq \frac{r}{2n_k}$.

Since $\bigcup \bar{\Delta}\left(0; \frac{r}{2n_{k+1}}, \frac{r}{2n_k}\right)$ cover a punctured neighborhood of 0, say $\tilde{\Delta}^*$, we have

$$|f(w)| \leq 2M \quad \text{in } \tilde{\Delta}^* \implies f \text{ has a removable}$$

singularity at 0. This contradicts the fact that the singularity is essential.

If $f_{n_k} \overset{l.u.}{\rightrightarrows} \infty$ then $1/f_{n_k} \overset{l.u.}{\rightrightarrows} 0$. By the argument above $1/f$ has a removable singularity $\Rightarrow$ f has at worst a pole, a contradiction.

Conclusion $f|_{\Delta^*(a,r)}$ omits at most one value $\forall r$

If 2 values are achieved finitely many times, shrink r & note that in $\Delta^*(a, r^{new})$ two values are omitted.

Conclusion $f|_{\Delta^*(a,r)}$ takes on all complex numbers ∞-many times, with at most one exception.