Math 220C - Lecture 18

May 7, 2021

Announcements

(1) Review – Friday, May 14

(2) Office Hours:

Wed, May 12, 4–5

Fri, May 14, 1–2

Mon, May 17, 3–4

(3) Qualifying Exam – Tuesday, May 18, 5–8

(4) No lecture/office hours on Wednesday, May 19

(5) Plenty of Practice Exams linked on website

(6) Gradescope

10 lectures left – Minicourse on Riemann Surfaces

First Goal – Introduction & basic properties

- So far, we have done complex analysis for domains $G \subseteq \mathbb{C}$ & studied holomorphic functions

- Many results carry over if we replace $G \subseteq \mathbb{C}$ by Riemann surfaces.

- The subject merges ideas from Complex Analysis with Geometry & Topology

- Connections w/ many fields

topology

differential geometry

algebraic geometry

arithmetic geometry

number theory

dynamics

....

Historically, Riemann Surfaces arose from attempts to understand analytic continuation of multi-valued functions

e.g. log ; algebraic functions

See Conway $\underline{\overline{IX}}$.

Riemann Surfaces – first defined by Riemann in his dissertation 1851

– the same dissertation considered the Riemann Mapping Theorem (Math 220B).

Bernhard Riemann (1826 – 1866).

Riemann surfaces were introduced by Riemann in his dissertation at Göttingen (1851). This transformed complex analysis, merging it with topology & algebraic geometry.

"We restrict the variables x, y to a finite domain by considering as the locus of the point O no longer the plane A but a surface T spread over the plane"

"We admit the possibility ... of covering the same part of the plane several times. However in such a case, we assume that those parts of the surface lying on top of one another are not connected by a line. Thus a fold or a splitting of parts of the surface cannot occur."

Translation by. R. Remmert,

"From Riemann surfaces to complex spaces"

Soc. Math. France, Congr 3 (1998)

- Klein : "Riemann's methods were regarded almost with distrust by other mathematicians".

- Ahlfors : "Riemann's writings are full of almost cryptic messages to the future".

§ 1. Sheaves

Sheaves in agriculture – a collection of stalks bundled together

Sheaves in mathematics

– we seek to formalize the concept of "function-like objects" e.g. holomorphic functions on Riemann surfaces

– the most elegant way of doing so is via sheaf theory

Definition Let X be a topological space. A presheaf of sets, abelian groups, rings ... is an assignment

$$U \rightsquigarrow \mathcal{F}(U)$$

of sets, abelian groups, rings... for all $U \subseteq X$ open.

& restriction maps

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

which should be homomorphisms of ... . We require

[i] $\rho_{UU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity

[ii] $\forall\ W \subseteq V \subseteq U$ we have

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"'] \arrow[rr, bend right, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"'] & \mathcal{F}(W) \end{tikzcd}$$

Terminology

[i] elements $s \in \mathcal{F}(U)$ are called sections.

[ii] restriction maps $\rho_{UV}(s) = s/_V$.

Definition A presheaf $\mathcal{F} \to X$ is a **sheaf** provided

$\forall$ $\mathcal{U} = \bigcup_i \mathcal{U}_i$ open cover, $s_i \in \mathcal{F}(\mathcal{U}_i)$ with

$$s_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$$

$\Rightarrow$ $\exists!$ $s \in \mathcal{F}(\mathcal{U})$ such that $s|_{\mathcal{U}_i} = s_i$.

Examples

[i] X topological space, $\mathcal{F} = \mathcal{C}$ is the sheaf:

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ continuous} \}.$$

with the usual restriction maps $\mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(V)$, $f \rightarrow f|_V$.

[ii] $X \subseteq \mathbb{R}^n$ open, $\mathcal{F} = \mathcal{C}^k$, $0 \leq k \leq \infty$, $k = \omega$

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ of class } \mathcal{C}^k \}$$

is a sheaf.

III $G \subseteq \mathbb{C}$ open, $\mathcal{F} = \mathcal{O}_G$, $\mathcal{U} \subseteq G$ open

$$\mathcal{O}_G(\mathcal{U}) = \{ f : \mathcal{U} \to \mathbb{C} \text{ holomorphic} \}$$ is a sheaf.

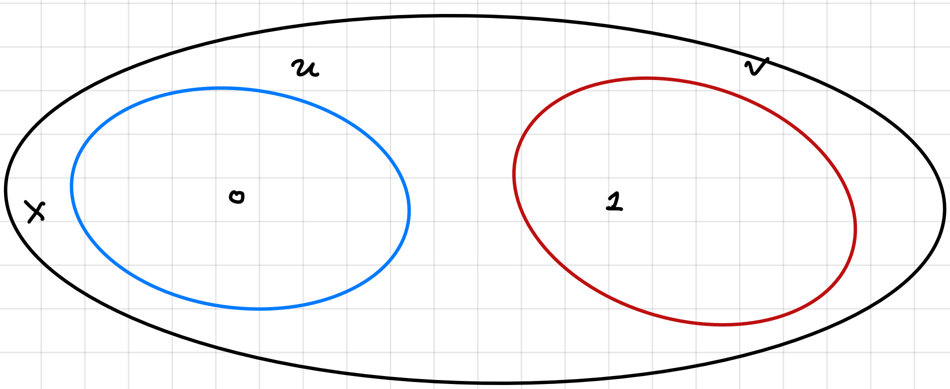
IV $p \in X$ topological space. The skyscraper sheaf

$$\mathbb{C}_p(\mathcal{U}) = \begin{cases} \mathbb{C} & \text{if } p \in \mathcal{U} \\ 0 & \text{if } p \notin \mathcal{U} \end{cases}$$

V The constant presheaf over X = top. space

$$\underline{\mathbb{C}}(\mathcal{U}) := \{ f : \mathcal{U} \to \mathbb{C} \text{ constant} \}$$ is not a sheaf.

Why? Assume $\mathcal{U}, V \subseteq X$



$\mathcal{U} \cap V = \emptyset$

$f_1 \equiv 0$ on $\mathcal{U}$

$f_2 \equiv 1$ on V

$\Rightarrow f_1|_{\mathcal{U} \cap V} = f_2|_{\mathcal{U} \cap V}$.

Let $W = \mathcal{U} \cup V$. Gluing fails.

However

$\underline{\mathbb{C}}^{sh} : \mathcal{U} \longrightarrow \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ locally constant} \}$ is a sheaf.

$\boxed{VII}$ Restriction of sheaves to open sets

$\mathcal{F} \longrightarrow X$ sheaf, $\mathcal{U} \subseteq X$ open

Define $\mathcal{F}/_{\mathcal{U}}$ a sheaf over $\mathcal{U}$ via

$\mathcal{F}/_{\mathcal{U}}(V) = \mathcal{F}(V)$ for $V \subseteq \mathcal{U}$ open. Note that

$V \subseteq X$ is also open since $\mathcal{U} \subseteq X$ is open, so the above makes sense.

Sheaves were discovered by Leray in the 40s as POW.

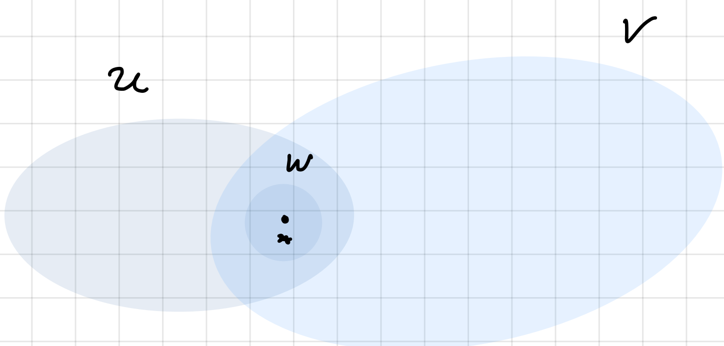
His papers were sent to Hopf in Zürich for publication.

Stalks & Germs

$\mathcal{F} \longrightarrow X$ presheaf, $x \in X$

Consider pairs (U, s), consisting of $x \in U \subseteq X$ open and $s \in \mathcal{F}(U)$ a section.

$(U, s) \sim (V, t)$ provided $\exists\, x \in W \subseteq U \cap V$ open with $s|_W = t|_W$.



This is an equivalence relation.

The stalk of $\mathcal{F}_x$ is the set of equivalence classes.

An equivalence class is called a germ.

We have $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$