Math 220C - Lecture 19

May 10, 2021

Goals

I. Define Riemann Surfaces

II. Define holomorphic functions

III. examples

Aside (Point Set Topology) X Hausdorff

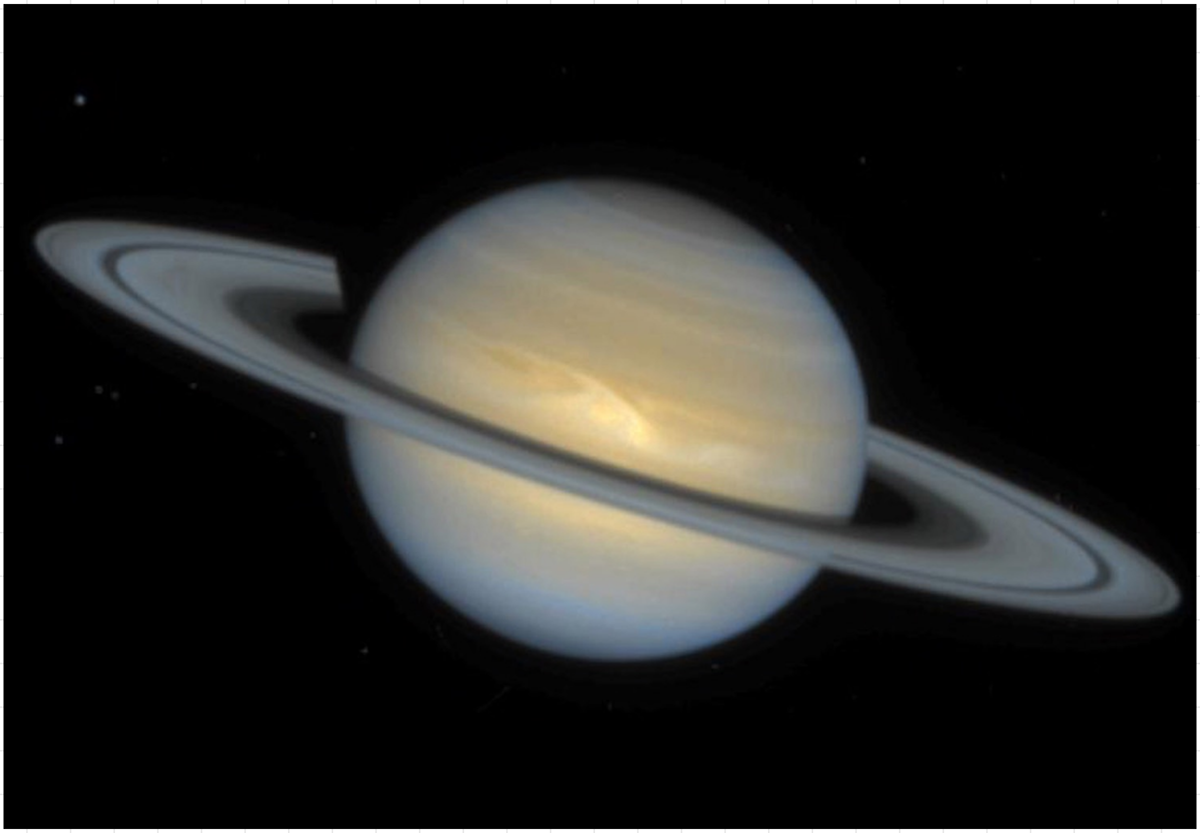
$\boxed{i}$ X is 2^{nd} countable if X admits a countable base for its topology

$\boxed{ii}$ X is paracompact if all open covers admit a locally finite subcover

$\boxed{iii}$ $X = \bigcup \mathcal{U}_\alpha$ open cover. A partition of unity $f_\alpha : X \longrightarrow \mathbb{R}$ continuous satisfies

- $\operatorname{supp} f_\alpha \subseteq \mathcal{U}_\alpha$ & $\operatorname{supp} f_\alpha$ is locally finite
- $\sum f_\alpha = 1$, $0 \leq f_\alpha \leq 1$.

In general $\boxed{ii} \iff \boxed{iii}$. $\boxed{i} \iff \boxed{ii}$ for manifolds.



Ringed spaces

A ringed space $(X, \mathcal{O}_X)$ is the datum of

[i] X topological space

[ii] sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras of complex valued continuous functions. ("regular functions")

Morphisms

$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces

[i] f continuous

[ii] $\forall\, U \subseteq Y$, $\varphi \in \mathcal{O}_Y(U)$. the pullback $\varphi \circ f : f^{-1}(U) \longrightarrow \mathbb{C}$ is a section of $\mathcal{O}_X(f^{-1}U)$.

Remark By [i], $f^{-1}U$ is open which is needed for [ii] to make sense.

Example $G, G' \subseteq \mathbb{C}$

$f : (G, \mathcal{O}_G) \longrightarrow (G', \mathcal{O}_{G'})$ is a morphism of ringed spaces

$\iff$ f holomorphic.

Why? $\Leftarrow$ If φ holomorphic in $U \subseteq G'$ & f holomorphic then $\varphi \circ f$ is holomorphic in $f^{-1}(U)$.

$\Rightarrow$ If f morphism, let $\varphi(z) = z$ holomorphic in $U = G'$

Then $\varphi \circ f = f$ is holomorphic by condition $\boxed{ii}$.

Remark We have the notion of an isomorphism.

Remark If X ringed space, $(X, \mathcal{O}_X)$.

$U \subseteq X$ open $\Rightarrow (U, \mathcal{O}_X|_U)$ is a ringed space.

Definition A $\mathscr{C}^k$-manifold ($k \geq 0$, $k = \infty$, $k = \omega$) of dim. n.

[i] X Hausdorff, connected, 2nd countable

[ii] $\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ and open subsets $G_\alpha \subseteq \mathbb{R}^n$ such that $(\mathcal{U}_\alpha, \mathcal{O}_X/_{\mathcal{U}_\alpha})$ is isomorphic as a ringed space to $(G_\alpha, \mathscr{C}^k)$.

Definition A Riemann surface $(X, \mathcal{O}_X)$ is

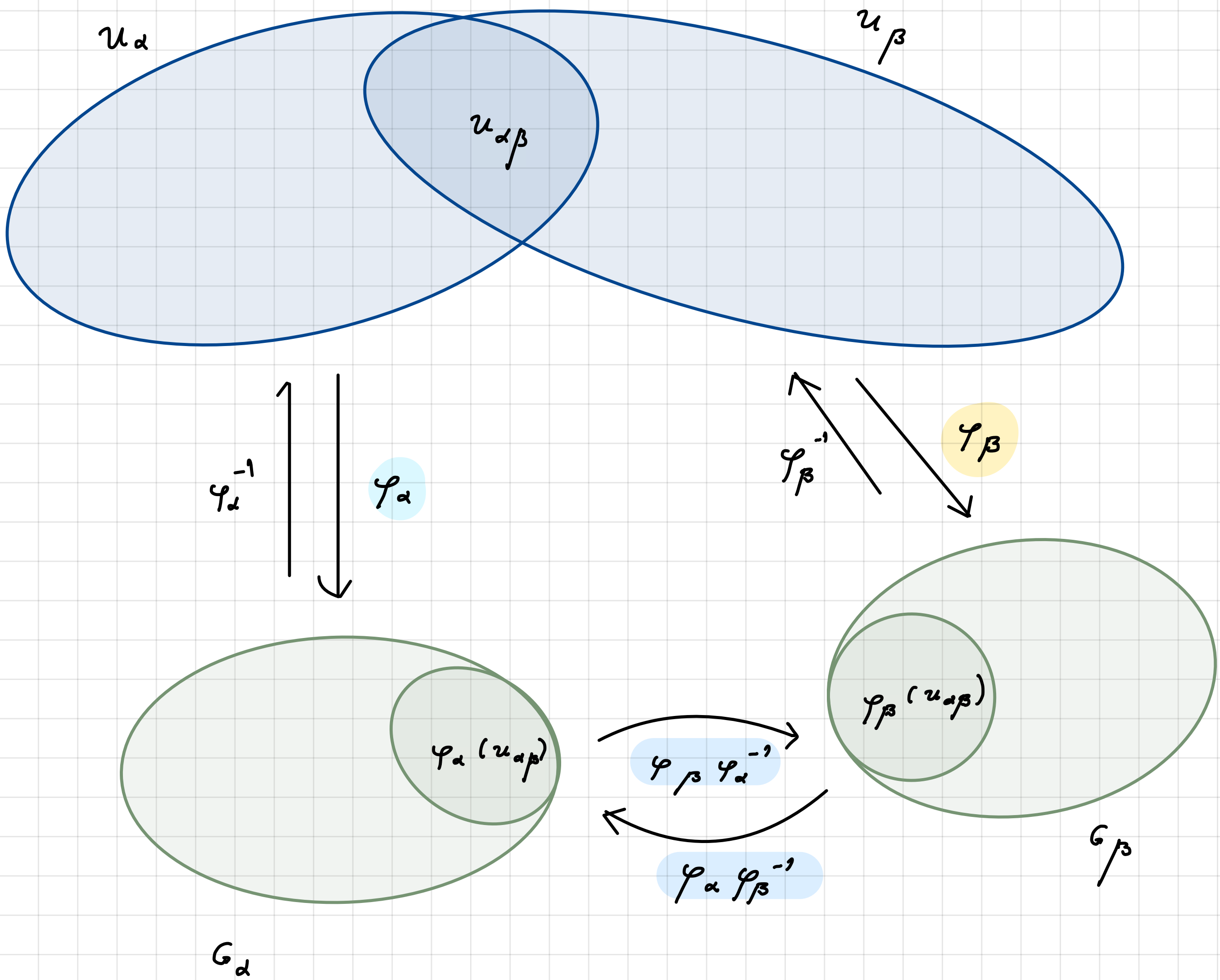
[i] X Hausdorff, connected, 2nd countable top space

[ii] $\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ and open subsets $G_\alpha \subseteq \mathbb{C}$ such that $(\mathcal{U}_\alpha, \mathcal{O}_X/_{\mathcal{U}_\alpha})$ is isomorphic$_{\mathbb{C}}$ as a ringed space to $(G_\alpha, \mathcal{O}_{G_\alpha})$.

Any Riemann surface is a $\mathscr{C}^k$-manifold of real dimension 2. $\forall k$.

In concrete terms Let X Riemann surface. Let $X = \bigcup_\alpha U_\alpha$

s.t. $(U_\alpha, \mathcal{O}_X|_{U_\alpha}) \cong (G_\alpha, \mathcal{O}_{G_\alpha})$ via isomorphism φ_α.



Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Note $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \longrightarrow \varphi_\beta(U_{\alpha\beta})$.

must be an isomorphism of ringed spaces.

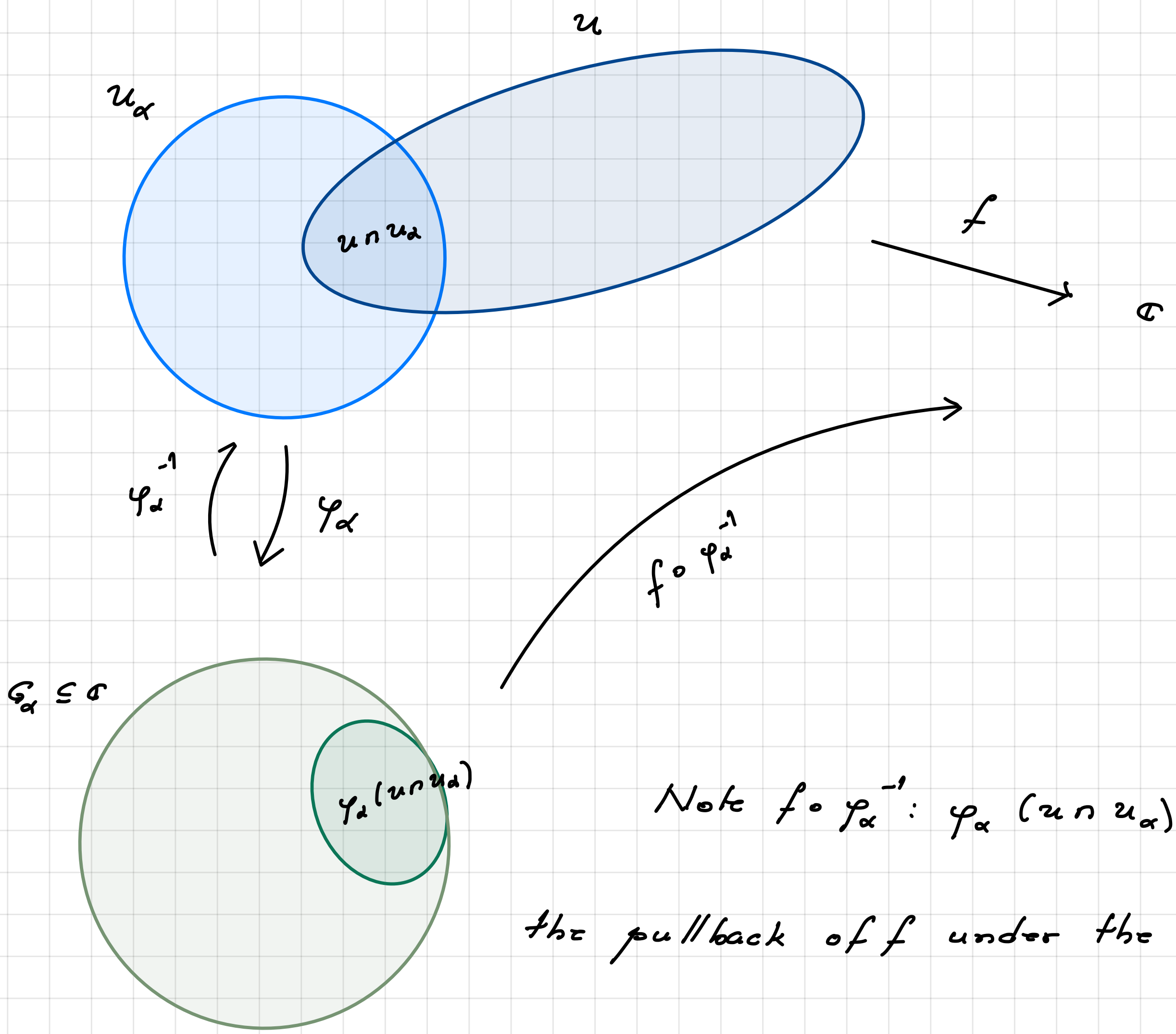
Thus $\varphi_\beta\varphi_\alpha^{-1}$ is a biholomorphism between open subsets of $\mathbb{C}$.

Holomorphic functions

Let X be a Riemann surface & $U \subseteq X$ open.

A holomorphic function on U is a section of $\mathcal{O}_X(U)$.

Concretely



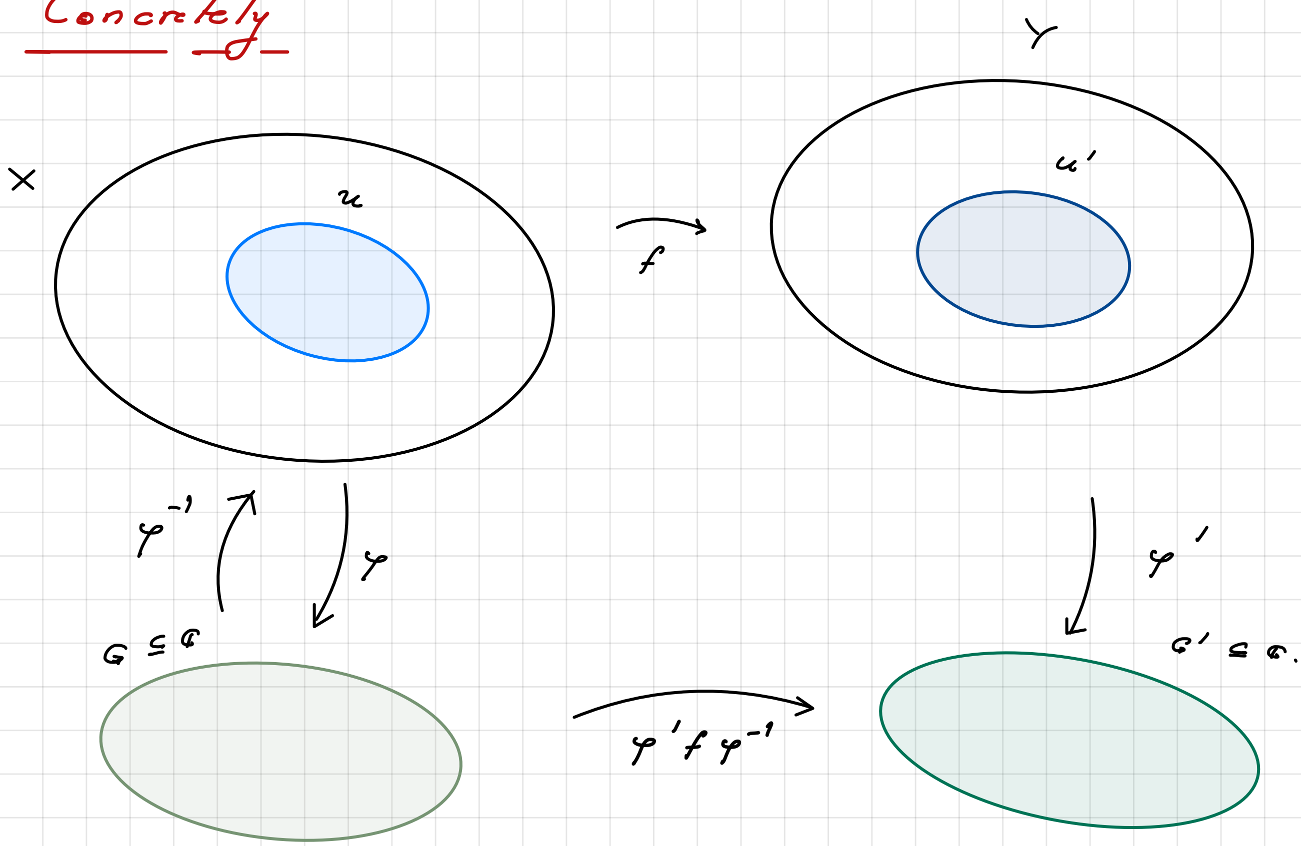
Note $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha) \longrightarrow \mathbb{C}$ is the pullback of f under the morphism φ_α^{-1}.

Therefore $f \circ \varphi_\alpha^{-1}$ is holomorphic in the set $\varphi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha) \subseteq \mathbb{C}$.

Holomorphic maps between Riemann Surfaces

$f : X \longrightarrow Y$ holomorphic iff f is a morphism of ringed spaces.

Concretely



If (U, G, φ) and (U', G', φ') are coordinate charts with $f(U) \subseteq U'$ we have

$$\varphi' f \varphi^{-1} : \varphi(U) \longrightarrow \varphi'(U') \text{ is a morphism of ringed}$$

spaces $\Rightarrow$ $\varphi' f \varphi^{-1}$ is holomorphic as a map between subsets of $\mathbb{C}$.