Math 220C - Lecture 2

March 31, 2021

Last time

Mean value Property

$$\forall a \in G, \ \bar{\Delta}(a,r) \subseteq G, \ u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt.$$

Maximum Principle

$u : G \longrightarrow \mathbb{R}$ continuous & MVP $\Rightarrow$

u cannot achieve a maximum (minimum) in G.

Notation $\partial_\infty G$ = extended boundary in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$$\partial_\infty G = \begin{cases} \partial G, & G \text{ bounded} \\ \partial G \cup \{\infty\}, & G \text{ unbounded} \end{cases}$$

A stronger version (MP^+)

(1) $u : G \longrightarrow \mathbb{R}$, u satisfies MVP in G, u continuous

(2) $\forall\, a \in \partial_\infty G$: $\displaystyle\limsup_{z \to a} u(z) \leq 0$.

Then either $u < 0$ in G or $u \equiv 0$ in G.

Proof We will show $u \leq 0$ in G. By the usual MP, u cannot have a maximum in G unless u = constant. This gives the statement we seek. Indeed, if $\exists\, \alpha \in G$ with $u(\alpha) = 0 \Rightarrow \alpha$ maximum in $G \Rightarrow u \equiv 0$. Else $u(\alpha) < 0$, $\forall\, \alpha \in G$.

Thus $u \equiv 0$ or $u < 0$ in G.

To show $u \le 0$, assume that $\exists\, \alpha \in G$ with $u(\alpha) > 0$.

Let $\varepsilon = u(\alpha) > 0$.

Let $K = \{ z \in G : u(z) \ge \varepsilon \}$. Since $\alpha \in K \Rightarrow K \ne \emptyset$.

Claim K is compact.

Assuming this, u cont., u will achieve a maximum in K at z_0. In particular $u(z_0) \ge \varepsilon$. Outside of K, $u < \varepsilon$. Thus z_0 will achieve a maximum for in u in G. This shows u constant.

Condition (2) ensures $u = \text{constant} \le 0$.

Proof of claim Let $z_n \in K$. We show that passing to a subseq. z_n converges in K. Note $z_n \in \hat{\mathbb{C}}$ & $\hat{\mathbb{C}}$ is compact. Thus WLOG we may assume $z_n \longrightarrow z \in \hat{\mathbb{C}}$ after passing to a subsequence.

Note $u(z_n) \ge \varepsilon$. If $z \in G \Rightarrow u(z) = \lim u(z_n) \ge \varepsilon. \Rightarrow z \in K$.

as needed. Else $z \in \partial_\infty G$. Then

$$\limsup_{z_n \to z} u(z_n) \ge \varepsilon \quad \text{which contradicts (2).}$$

Thus K is compact.

Corollary G bounded, $u : \overline{G} \longrightarrow \mathbb{R}$ cont., MVP,

$u \equiv 0$ on ∂G $\Rightarrow$ $u \equiv 0$ in G.

Proof We use ME^{+}. We need to verify condition (2).

G bounded, $\partial_\infty G = \partial G$. If $a \in \partial G$, $\lim\limits_{z \to a} u(z) = \underbrace{u(a)}_{\text{continuity in } \overline{G}} = 0$.

Thus $u < 0$ in G or $u \equiv 0$ in G.

Argue in the same way for $-u$. $\Rightarrow$ either $-u < 0$ in G or $-u \equiv 0$ in G. Thus $u \equiv 0$ in G.

Remark $u, v : \overline{G} \longrightarrow \mathbb{R}$ continuous & harmonic in G. & G bounded. If

$$u/_{\partial G} = v/_{\partial G} \Rightarrow u = v \text{ in } G.$$

Thus $u/_{\partial G} \rightsquigarrow u$ in G uniquely.

§2. Poisson Formula & Dirichlet Problem

Question 1 $u : \overline{G} \longrightarrow \mathbb{R}$ continuous, harmonic in G, G bounded.

$$u\big/_{\partial G} \rightsquigarrow u \quad \text{uniquely in } G.$$

Find a formula for u in G, from the values $u/_{\partial G}$.

We will solve this for $G = \Delta(0,1)$. or $\Delta(a, R)$. $\rightsquigarrow$ Poisson Formula

Question 2 Given $f : \partial G \longrightarrow \mathbb{R}$ continuous, is there

$u : \overline{G} \longrightarrow \mathbb{R}$ continuous and

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \\ u\big/_{\partial G} = f \end{cases}$$

Dirichlet Problem

(boundary value problem)

Harmonic Functions on the unit disc $\Delta = \Delta(0,1)$

Given $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ,

find a formula for $u(a)$ in terms of $u\big|_{\partial\Delta}$.

Remark $a = 0$ Use MVP over the circle $|z| = r$, $r < 1$.

This smaller circle is contained in Δ, where u satisfies MVP.

then

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt$$

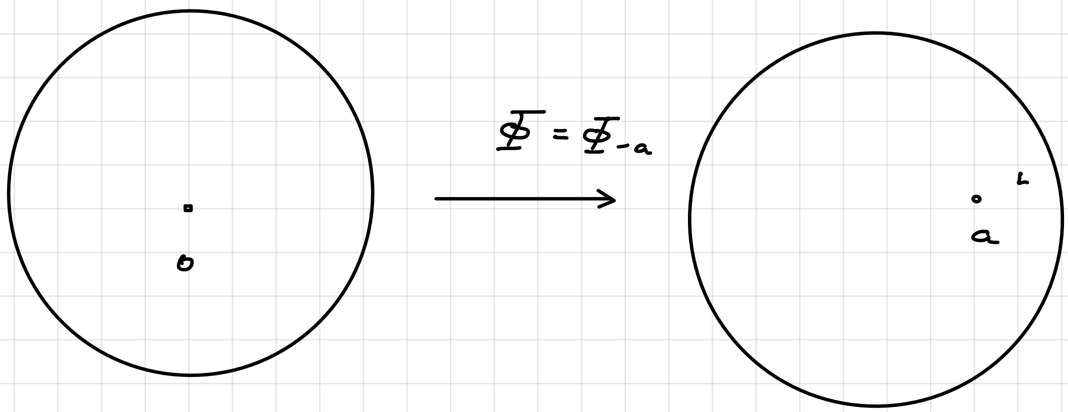
Since u continuous over $\overline{\Delta}$, make $r \longrightarrow 1$. This yields

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\, dt.$$

(To justify the limit use that $u(re^{it}) \longrightarrow u(e^{it})$ uniformly since u is uniformly cont. over $\overline{\Delta}$).

Question: How about the case $a \neq 0$?

General Case



Idea : Recenter! $\quad \Phi : \Delta \longrightarrow \Delta, \ \partial\Delta \longrightarrow \partial\Delta$

$$\Phi(z) = \frac{z+a}{1+\bar{a}z}, \quad \Phi(0) = a.$$

Then $\tilde{u} = u \circ \Phi : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ (Problem 1, HWK1)

Apply MVP to $\tilde{u}$

$$u(a) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{is})\, ds = \frac{1}{2\pi}\int_0^{2\pi} u(\Phi(e^{is}))\, ds.$$

Since $\Phi(e^{is}) \in \partial\Delta$ this also shows $u(a)$ is given explicitly in terms of $u|_{\partial\Delta}$.

Next time : We will work out a more explicit expression

$\Rightarrow$ Poisson Integral Formula

Slogan

MVP + Aut Δ $\Rightarrow$ Poisson's formula