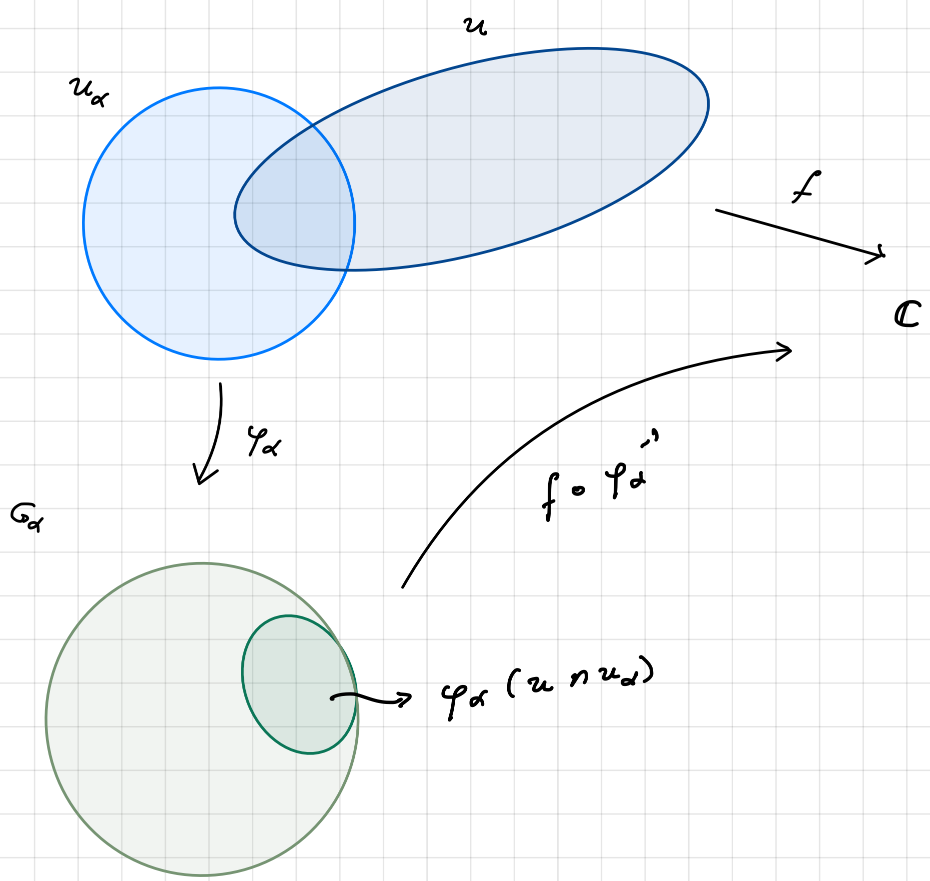
Math 220C - Lecture 20

May 12, 2021

Holomorphic functions

Let X be a Riemann surface, $(U_\alpha, G_\alpha, \varphi_\alpha)$ coordinate charts.



We showed last time that

f holomorphic iff $f \circ \varphi_\alpha^{-1}$ is holomorphic in $\varphi_\alpha(U \cap U_\alpha)$ $\forall \alpha$.

Remark We can also turn this discussion around.

Let X be a topological space (Hausdorff, 2^{nd} countable)

$X = \bigcup_\alpha U_\alpha$ open cover. Assume we are given

- $\varphi_\alpha : U_\alpha \longrightarrow G_\alpha$ homeomorphisms, $G_\alpha \subseteq \mathbb{C}$ such that

- $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ biholomorphism

These are called compatible coordinate charts

Then X becomes a Riemann surface.

Issue Define the sheaf $\mathcal{O}_X$.

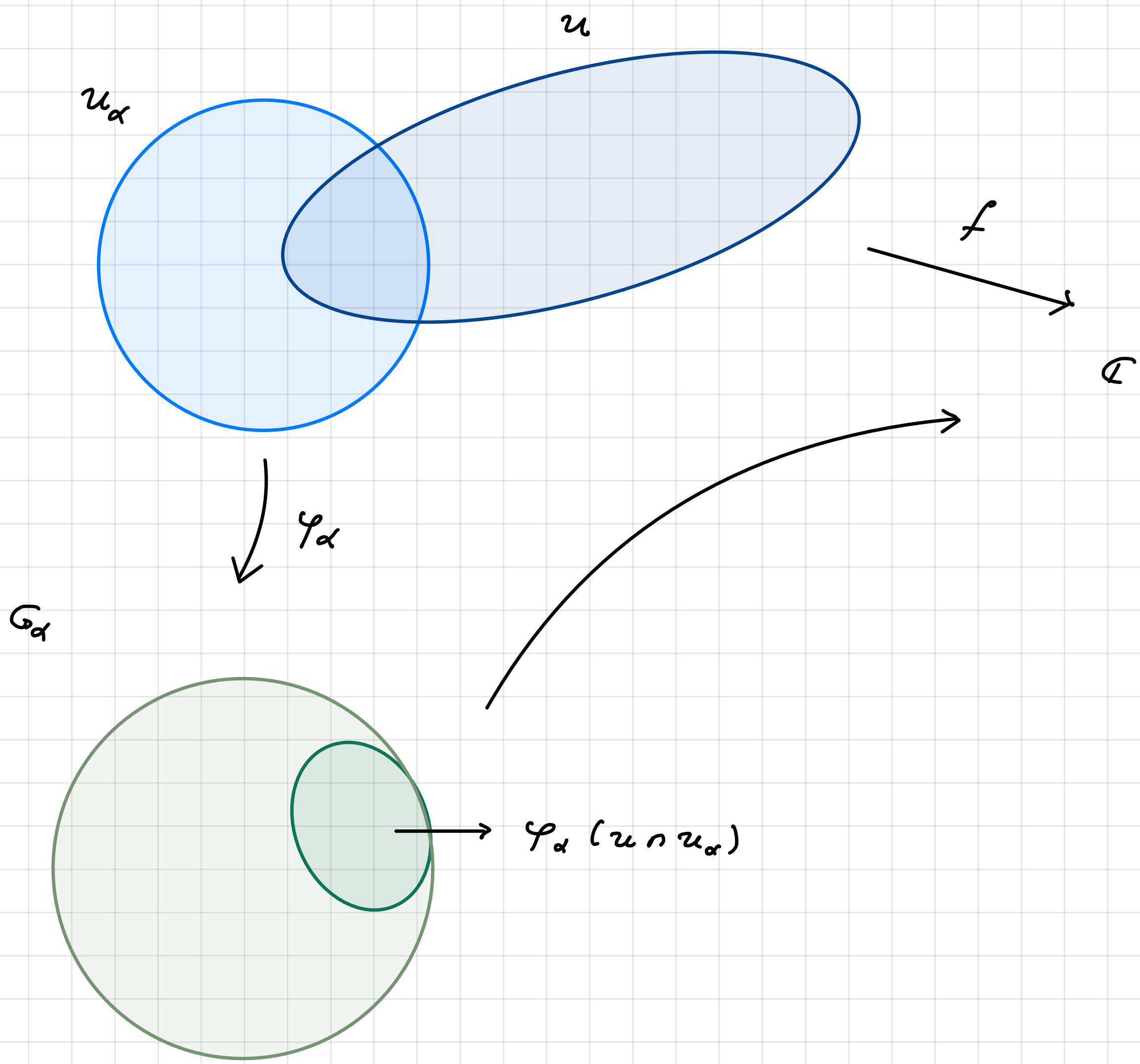
Note U open $\iff$ $U \cap U_\alpha$ open $\iff$ $\varphi_\alpha(U \cap U_\alpha)$ open in G_α.

Declare $f : U \longrightarrow \mathbb{C}$ to be a section of $\mathcal{O}_X$ provided.

$$f \in \mathcal{O}_X(U) \iff f \circ \varphi_\alpha^{-1} \text{ holomorphic in } \varphi_\alpha(U \cap U_\alpha).\ \forall \alpha.$$

Check $\mathcal{O}_X$ is indeed a sheaf. & $(X, \mathcal{O}_X)$ Riemann surface

Meromorphic functions



Definition

f meromorphic in $\mathcal{U}$ provided $f\varphi_\alpha^{-1}$ meromorphic in $\varphi_\alpha(\mathcal{U}\cap\mathcal{U}_\alpha)$ $\forall\alpha$

Note there exists a sheaf $\mathcal{M}$ of meromorphic functions

$$\mathcal{U} \longrightarrow \text{meromorphic functions in } \mathcal{U}$$

Zeroes, poles, order

We define the order of a pole or a zero for f to be the order of a pole or a zero for $f \varphi_\alpha^{-1}$ at $\varphi_\alpha(p)$ for $p \in \mathcal{U}_\alpha$

Claim This is independent of choice of α.

Subclaim

Let g be meromorphic in $\mathcal{U}$, $a \in \mathcal{U}$. Let $T : V \longrightarrow \mathcal{U}$ be a biholomorphism with $T(b) = a$, $b \in V$. Then

g has order m at a $\Rightarrow$ $g \circ T$ has order m at b.

We use this for $g = f \varphi_\alpha^{-1}$, $a = \varphi_\alpha(p)$

$\Rightarrow g \circ T = f \varphi_\beta^{-1}$.

$T = \varphi_\alpha \varphi_\beta^{-1}$, $b = \varphi_\beta(p)$.

The subclaim shows that the order thus defined is independent of the choice of α.

Proof of the Subclaim

WLOG $a = b = 0$, else we can translate.

Write $g(z) = z^m G(z)$, $G(0) \neq 0$.

Since $T(0) = 0$ & $T'(0) \neq 0$ since T is biholomorphism, we have $T(z) = z\, S(z)$, $S(0) \neq 0$.

Note
$$g \circ T(z) = T(z)^m G(T(z))$$
$$= z^m S(z)^m G(T(z)).$$

Since $S(z)^m G(T(z))\big|_{z=0} = S(0)^m G(0) \neq 0 \Rightarrow$

$\Rightarrow$ order $g \circ T$ at $z = 0$ equals m. as needed.

Remarks Essential singularities can be defined similarly.

Aside – Divisors on Riemann surfaces

Definition A divisor on a Riemann surface X is a formal sum

$$D = \sum_{p \in X} n_p [p] \quad \text{with } n_p \in \mathbb{Z} \text{ such that}$$

$S = \{ p : n_p \neq 0 \}$ is locally finite.

Examples

[I] $X = \hat{\mathbb{C}}$, $D = 2[0] + 3[\infty] - 5[1]$ divisor on X

[II] D is said to be effective if $n_p \geq 0 \ \forall p \in X$

[III] Divisors can be formally added & subtracted

$$D = \sum n_p [p], \quad E = \sum m_p [p]$$

$$\Longrightarrow \quad D \pm E = \sum (n_p \pm m_p) [p] \quad \text{is a divisor}$$

[IV] restrictions, $U \subseteq X$ open. If

$$D = \sum_{p \in X} n_p [p] \Longrightarrow D/_U = \sum_{p \in U} n_p [p]$$

[IV] $\exists$ sheaf of divisors $\underline{\mathcal{D}iv}$.

$$U \longrightarrow \{ \text{divisors in } U \}$$

[V] degree. If X is compact, any divisor is a finite sum.

$$D = \sum n_p [p], \quad n_p \in \mathbb{Z}. \implies \deg D := \sum_p n_p.$$

Principal divisors If f meromorphic in X, define

[I] $$\operatorname{div} f = \sum_{z \in X} \operatorname{ord}(f, z) [z]$$

$$= \sum_{z \text{ zero}} \operatorname{mult}_z(f) [z] - \sum_{p \text{ pole}} \operatorname{mult}_p(f) [p]$$

[II] Check: $\operatorname{div}(fg) = \operatorname{div} f + \operatorname{div} g$.

Example $X = \hat{\mathbb{C}}$. $f = \dfrac{\prod_{i=1}^{m} (z - a_i)}{\prod_{i=1}^{n} (z - b_i)}$ meromorphic function in $\hat{\mathbb{C}}$

$a_i, b_i \in \mathbb{C}$.

$$\operatorname{div} f = \sum_{i=1}^{m} [a_i] - \sum_{i=1}^{n} [b_i] + (n-m) [\infty]$$

$$\implies \deg \operatorname{div} f = \sum_{i=1}^{m} 1 - \sum_{i=1}^{n} 1 + (n-m) = 0.$$

Examples of Riemann surfaces

$\boxed{I}$ not compact

$\boxed{II}$ compact

Non-compact examples

$\boxed{a}$ $G \subseteq \mathbb{C}$ open subset is a Riemann surface

$\boxed{b}$ $X \subseteq \mathbb{C}^2$, $X = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\} \subseteq \mathbb{C}^2$.

Assume $\forall p \in X$,

$$f_x(p) \neq 0 \quad \text{or} \quad f_y(p) \neq 0.$$

Claim X is a Riemann surface

Proof We construct charts & show they are compatible.

Let $p \in X$.

- if $f_y(p) \neq 0 \Rightarrow$ by implicit function theorem,

$\exists\, p \in \mathcal{U} \subseteq X$ open such that

$y = g(x)$ for $(x,y) \in \mathcal{U}$ where $g : V \longrightarrow \mathbb{C}$ is holomorphic.

Then $U \longrightarrow G$, $(x,y) \longrightarrow x$ has inverse

$x \longrightarrow (x, g(x))$. $\Rightarrow U$ is a chart

- If $f_x(p) \neq 0$, we similarly have

$x = h(y)$ for $(x,y) \in U$, $h: H \longrightarrow \mathbb{C}$ holomorphic

Then $U \longrightarrow H$ is a chart $(x,y) \longrightarrow y$ with inverse

$y \longrightarrow (h(y), y)$. $\Rightarrow U$ is a chart

Compatibility Charts of the first type are clearly compatible. Same for charts of 2^{nd} type.

We check compatibility between charts of different types.

WLOG we may assume we are around a point p with

$f_x(p) \neq 0$ & $f_y(p) \neq 0$.

Then the change of coordinates is

$$\begin{tikzcd} & \mathcal{U} \arrow[r] \arrow[dl] & G \arrow[dll, "T"'] \\ H & & \end{tikzcd}$$

$$\begin{tikzcd} & (x,y) \arrow[r] \arrow[dl] & x \arrow[dll, "T"'] \\ y & & \end{tikzcd}$$

$$T : G \longrightarrow H$$
$$x \longrightarrow y = g(x)$$

$$T^{-1} : H \longrightarrow G$$
$$y \longrightarrow x = h(y)$$

Both T & T^{-1} are holomorphic, as needed.