Math 220C - Lecture 21

May 21, 2021

Last time

- noncompact Riemann surfaces

$X \subseteq \mathbb{C}$ or $X = \{ f(z,w) = 0 \} \subseteq \mathbb{C}^2$

Require: $\partial_z f(p) \neq 0$ or $\partial_w f(p) \neq 0$ if $p \in X$.

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§1. Compact Riemann Surfaces

a $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

We construct charts $\mathcal{U}_0, \mathcal{U}_\infty$, $X = \mathcal{U}_0 \cup \mathcal{U}_\infty$

$$\mathcal{U}_0 = \{ z : z \neq \infty \} \xrightarrow{\phi_0} \mathbb{C} \quad , \quad z \longrightarrow z$$

$$\mathcal{U}_\infty = \{ z : z \neq 0 \} \xrightarrow{\phi_\infty} \mathbb{C} \quad , \quad z \longrightarrow \frac{1}{z}$$

These two charts are compatible. The transition map is

$$T = \phi_\infty \phi_0^{-1} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \; , \; z \longrightarrow 1/z \quad \text{biholomorphic}$$

$\Rightarrow X$ Riemann surface

Projective curves

Let $\mathbb{P}^2 = \{ [x : y : z] \,,\ (x, y, z) \neq (0,0,0) \,,\ x, y, z \in \mathbb{C} \} / \sim$

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad \text{if } \lambda \in \mathbb{C}^\times.$$

$$U_1 = \{ x \neq 0 \} \xrightarrow{\phi_1} \mathbb{C}^2 \qquad [x : y : z] \longrightarrow \left(\frac{y}{x}, \frac{z}{x} \right).$$

$$U_2 = \{ y \neq 0 \} \xrightarrow{\phi_2} \mathbb{C}^2 \qquad [x : y : z] \longrightarrow \left(\frac{x}{y}, \frac{z}{y} \right).$$

$$U_3 = \{ z \neq 0 \} \xrightarrow{\phi_3} \mathbb{C}^2 \qquad [x : y : z] \longrightarrow \left(\frac{x}{z}, \frac{y}{z} \right).$$

Let f homogeneous of degree d. in variables x, y, z.

(*) if $p \in \mathbb{P}^2$, $f(p) = 0$ then $f_x(p) \neq 0$ or $f_y(p) \neq 0$ or $f_z(p) \neq 0$.

Then

$$X = \{ [x : y : z] : f(x, y, z) = 0 \} \hookrightarrow \mathbb{P}^2$$

is a Riemann Surface (check).

$\boxed{C}$ torus : $\omega_1, \omega_2 \neq 0$, $\omega_1/\omega_2 \notin \mathbb{R}$

Let. $\Lambda = \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \} \hookrightarrow \mathbb{C}$

Let $X = \mathbb{C}/\Lambda$ where Λ acts on $\mathbb{C}$ by translations

Let $\pi : \mathbb{C} \longrightarrow X$

$\boxed{i}$ X has the quotient topology, $\mathcal{U} \subseteq X$:

$$\mathcal{U} \text{ open} \iff \pi^{-1}\mathcal{U} \text{ open}$$

$\boxed{ii}$ π continuous & open

$\mathcal{U}$ open, $\pi(\mathcal{U})$ open since $\pi^{-1}\pi\,\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{U} + \lambda$ = open.

$\boxed{iii}$ coordinate charts. Let $\varepsilon < \frac{1}{2} \min_{\lambda \in \Lambda} |\lambda|$.

Let $x \in X$, $\pi(z) = x$, $z \in \mathbb{C}$

Let $\pi : \Delta(z, \varepsilon) \longrightarrow D'_{z,\varepsilon} = \pi(\Delta(z,\varepsilon))$.

Claim $D_{z,\varepsilon}$ is a chart

π surjective, injective, continuous, open hence a homeomorphism

Claim The charts $D_{z,\varepsilon}$ are compatible

$\varphi_1 : D_{z_1,\varepsilon} \longrightarrow \Delta(z_1, \varepsilon)$

$\varphi_2 : D_{z_2,\varepsilon} \longrightarrow \Delta(z_2, \varepsilon)$

$U = D_{z_1,\varepsilon} \cap D_{z_2,\varepsilon}$

$T = \varphi_2 \varphi_1^{-1}$ is given by $z \longrightarrow z + \lambda$. on $\varphi_1(U)$.

Indeed $\pi T(z) = \pi \varphi_2 \varphi_1^{-1}(z) = \pi \varphi_2 \pi(z) = \pi(z)$

$\Rightarrow T(z) = z + \lambda$. biholomorphic

Conclusion Give X the complex structure determined by these charts. X is a Riemann surface.

Remark Meromorphic functions on $X = \mathbb{C}/\Lambda$ are meromorphic functions in $\mathbb{C}$, Λ-periodic

$\rightsquigarrow$ elliptic functions e.g. $\wp, \wp', \wp'', \ldots$

§2. Basic Results

[a] Identity Theorem

$f, g : X \longrightarrow Y$ holomorphic maps between Riemann Surf.

$S = \{ x : f(x) = g(x) \}$ has a limit point in X.

Then $f \equiv g$.

[b] Open Mapping theorem

$f : X \longrightarrow Y$ holomorphic, non constant $\Rightarrow$ f is open

[c] Maximum Modulus

$f : X \longrightarrow \mathbb{C}$ holomorphic & $|f|$ has a maximum at $p \in X$

$\Rightarrow$ f constant.

Corollary

$f : X \longrightarrow \mathbb{C}$, X compact $\Rightarrow$ f constant

Proof of Maximum Principle

Let $p \in U_\alpha$. Let $\varphi_\alpha : U_\alpha \longrightarrow G_\alpha$ be a chart. Let

$f \circ \varphi_\alpha^{-1} : G_\alpha \longrightarrow \mathbb{C}$, $G_\alpha \subseteq \mathbb{C}$. Then $|f \circ \varphi_\alpha^{-1}|$ has a maximum

at $\varphi_\alpha(p)$. By the usual maximum principle for G_α

$\Rightarrow f \circ \varphi_\alpha^{-1} = \text{constant}$ in G_α $\Rightarrow f = \text{constant}$ in U_α $\Rightarrow$

$\Rightarrow f = \text{constant}$ by the identity theorem.

Rephrasing in terms of sheaves

- $\mathcal{F} \longrightarrow X$, $H^0(X, \mathcal{F}) := \mathcal{F}(X)$

- X compact $\Rightarrow H^0(X, \mathcal{O}_X) = \mathbb{C}$.

Proof of Identity Principle

$$\Omega = \{ x \in X,\ f = g \text{ in a neighborhood of } x \}$$

Claims

$\boxed{i}$ $\Omega \neq \emptyset$

$\boxed{ii}$ Ω open $\implies \Omega = X \implies f \equiv g$.

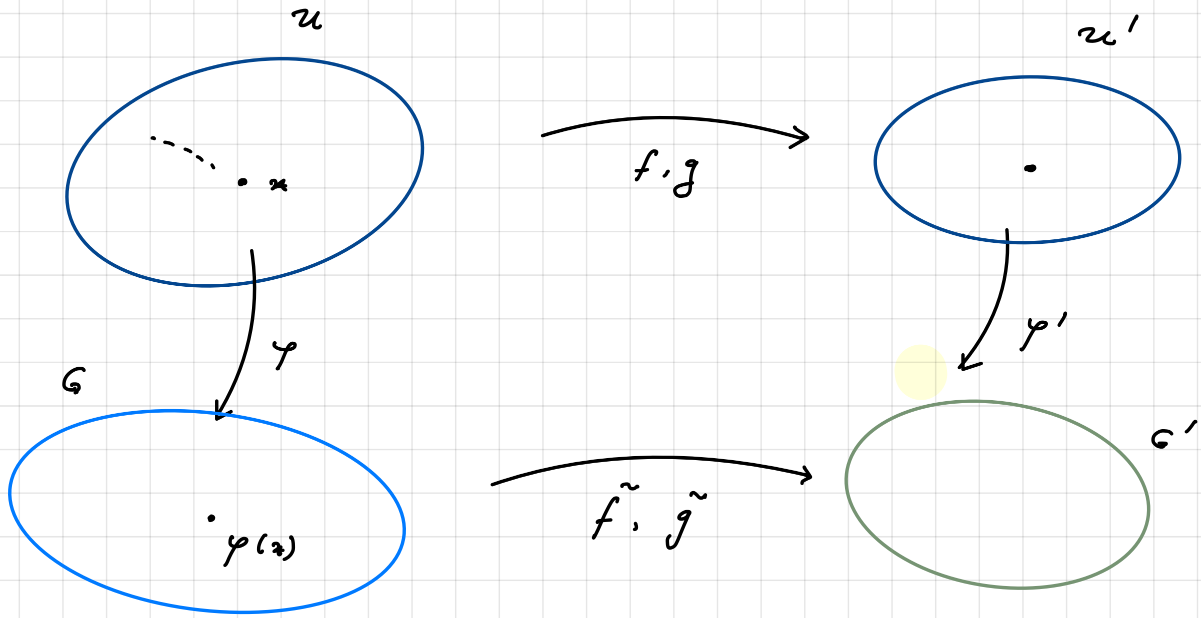
$\boxed{iii}$ Ω closed

Proof of $\boxed{i}$

Let $S = \{ s : f(s) = g(s) \} \subseteq X$ have a limit point x. We show $x \in \Omega$.

Let $\mathcal{U}$ be a chart near x, $\mathcal{U}'$ a chart in Y near $f(x) = g(x) = y$. Shrinking if needed we may assume

$f(\mathcal{U}) \subseteq \mathcal{U}'$, $g(\mathcal{U}) \subseteq \mathcal{U}'$, $\mathcal{U}$ connected.



Let $\varphi : \mathcal{U} \longrightarrow G$, $\varphi' : \mathcal{U}' \longrightarrow G'$ be coordinate charts, $G, G' \subseteq \mathbb{C}$. Let $\tilde{f} = \varphi' f \varphi^{-1}$; $\tilde{g} = \varphi' g \varphi^{-1}$. Let $\tilde{S} = \{\tilde{s} \in G : \tilde{f}(\tilde{s}) = \tilde{g}(\tilde{s})\}$. Note $\tilde{S} \supseteq \varphi(S)$ so $\tilde{S}$ has a limit point $\varphi(x) \Rightarrow \tilde{f} \equiv \tilde{g}$ in $G \Rightarrow f = g$ in $\mathcal{U} \Rightarrow$ $\Rightarrow x \in \Omega \Rightarrow \Omega \neq \emptyset$.

Part $\boxed{\text{II}}$ is clear by definition. Part $\boxed{\text{III}}$ is a repetition of the above argument. (check it!)

§3. Questions about functions on Riemann surfaces

Question A Is every divisor $D = \sum_{p \in X} n_p [p]$ the divisor of a meromorphic function?

Answer depends on X.

$\boxed{1}$ non-compact $X \subseteq \mathbb{C}$ open

If $D \geq 0$, $n_p \geq 0 \ \forall p$, the question is equivalent to the Weierstraß Problem.

In general write $D = D_+ - D_-$, D_+, D_- effective.

Write $D_+ = \operatorname{div} f_+$, $D_- = \operatorname{div} f_-$, $f = f_+/f_-$. Then

$$D = \operatorname{div} f_+ - \operatorname{div} f_- = \operatorname{div} f_+/f_- = \operatorname{div} f.$$

12 compact X

Example $X = \hat{\mathbb{C}}$. We need $\deg D = 0$ since we already noted $\deg \operatorname{div} f = 0$.

Conversely if $\deg D = 0$, $D = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

If $a_i, b_i \in \mathbb{C}$, let $f = \dfrac{\prod_{i=1}^{n} (z - a_i)}{\prod_{i=1}^{n} (z - b_i)} \Rightarrow \operatorname{div} f = D.$

If one of the a_i's or b_i's equals ∞, use first a FLT to reduce to the previous case. Thus

$$D \text{ principal} \iff \deg D = 0 \quad \text{if } X = \hat{\mathbb{C}}.$$

Example $X = \mathbb{C}/\Lambda$

Let $D = \sum_i z_i - \sum_i p_i$ be a divisor on X.

We allow repetitions among the z_i, p_i's.

Meromorphic functions on X are elliptic functions

We want $\operatorname{div} f = D \iff f$ has zeros / poles at z_i / p_i.

We have seen in Math 220A, Lecture 22 that

(1) $\#$ zeros $(f) = \#$ poles (f) $\iff$ $\deg D = 0$

(2) $\sum$ zeros of f $- \sum$ poles of $D \in \Lambda$

$\rightsquigarrow$ new condition

for any elliptic functions.

These were consequences of the argument principle.

Conversely, recall

$$\sigma(z) = z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} E_2\left(\frac{z}{\lambda}\right)$$ – Math 220B, HWK 2

$$f(z) = \frac{\prod_i \sigma(z - z_i)}{\prod_i \sigma(z - p_i)}$$

Issue (Check!) Using (1) & (2) one checks that f is Λ-periodic $\Rightarrow$ f elliptic function $\Rightarrow$ f is meromorphic function on X.

Note $\operatorname{div} f = \sum z_i - \sum p_i = D$.

Thus Question A $\Longleftrightarrow$ Conditions (1) & (2) for $X = \mathbb{C}/\Lambda$.