Math 220C - Lecture 22

May 24, 2021

Let X be a Riemann surface

Let $D = \sum_{p \in X} n_p [p]$ divisor on X.

Question A

Is every divisor D the divisor of a meromorphic function?

Answer depends on X.

[i] non-compact $X \subseteq \mathbb{C}$ open

Yes! Weierstraß Problem

If X compact — additional conditions are needed.

[ii] $X = \hat{\mathbb{C}}$ we need $\deg D = 0$

[iii] $X = \mathbb{C}/\Lambda$ we need $\deg D = 0$ & another condition

What is the issue ?

Cover X by coordinate charts $\mathcal{U}_\alpha$ with $\mathcal{U}_\alpha \overset{\varphi_\alpha}{\cong} G_\alpha \subseteq \mathbb{C}$.

Since we can solve the Weierstraß problem in $\mathcal{U}_\alpha$, we have

$$D\big/_{\mathcal{U}_\alpha} = \operatorname{div} f_\alpha \, , \quad f_\alpha \text{ meromorphic in } \mathcal{U}_\alpha .$$

Compatibility

$$\operatorname{div} f_\alpha \big/_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} = \operatorname{div} f_\beta \big/_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} = D\big/_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \implies$$

$$\implies \operatorname{div} f_\alpha / f_\beta = 0 \text{ in } \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

$\implies$ f_α / f_β is nowhere zero holomorphic in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

Want

f meromorphic in X, $\operatorname{div} f = D$

$$\iff \operatorname{div} f\big/_{\mathcal{U}_\alpha} = D\big/_{\mathcal{U}_\alpha} = \operatorname{div} f_\alpha$$

$$\iff \operatorname{div} f / f_\alpha = 0 \text{ in } \mathcal{U}_\alpha$$

$\iff$ f / f_α is nowhere zero holomorphic in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

Question A (rephrased) Given

- open cover $X = \bigcup \mathcal{U}_\alpha$, $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$
- $f_\alpha \in \mathcal{M}^*(\mathcal{U}_\alpha)$ with $f_\alpha / f_\beta \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$

we want $f \in \mathcal{M}^*(X)$ with

$$f / f_\alpha \in \mathcal{O}^*(\mathcal{U}_\alpha).$$

Notation

- $\mathcal{O}$ = sheaf of holomorphic functions
- $\mathcal{O}^*$ = sheaf of holomorphic nonvanishing fns.
- $\mathcal{M}$ = sheaf of meromorphic functions
- $\mathcal{M}^*$ = nonzero meromorphic functions

Aside : There is a similar additive question.

Question B

Given

$$X = \bigcup \mathcal{U}_\alpha \ , \ f_\alpha \text{ meromorphic in } \mathcal{U}_\alpha$$

such that

$$f_\alpha - f_\beta \in \mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

Want f meromorphic in X with

$$f - f_\alpha \in \mathcal{O}(\mathcal{U}_\alpha).$$

Special case

- $X \subseteq \mathbb{C}$ open, $\mathcal{U}_\alpha$ open near p_α, $p_\beta \notin \mathcal{U}_\alpha$ for $\alpha \neq \beta$
- f_α = Laurent principal part near p_α.

This recovers Mittag-Leffler.

Rephrasing in terms of sheaves

- $\mathcal{M}^*$ = sheaf of meromorphic functions $\neq 0$
- $\underline{\mathcal{D}iv}$ = sheaf of divisors

Recall $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

Rephrasing Question A

Is
$$\begin{aligned} H^0(X, \mathcal{M}^*) &\longrightarrow H^0(X, \underline{\mathcal{D}iv}) \\ f &\longmapsto \operatorname{div} f \end{aligned}$$
surjective?

Strategy for Answering Question A

We rephrase the problem using sheaf cohomology.

Goals

$\boxed{\text{I}}$ Cohomology

For all $\mathcal{F} \to X$, we will define $H^p(X, \mathcal{F})$, $p \geq 0$. We already have seen $H^0(X, \mathcal{F})$.

$\boxed{\text{II}}$ Exact sequences of sheaves

We will define
- morphisms of sheaves
- exact sequences of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

III Short exact sequences & cohomology

Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ we will show

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \longrightarrow$$

$$\hookrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow \cdots$$

We assume this for now & give the details starting next time.

How is this relevant for us?

[a] We will show that

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \xrightarrow{\mathrm{div}} \underline{\mathcal{D}iv} \longrightarrow 0 \quad \text{is exact.}$$

[b] If $H^1(X, \mathcal{O}^*) = 0$ $\Rightarrow$ Question A YES

Indeed,
$$H^0(X, \mathcal{M}^*) \xrightarrow{\mathrm{div}} H^0(X, \underline{\mathcal{D}iv}) \longrightarrow \underbrace{H^1(X, \mathcal{O}^*)}_{0}$$

$\Rightarrow$ div is surjective

[c] In the additive case, $\mathcal{O}^*$ gets replaced by $\mathcal{O}$.

If $H^1(X, \mathcal{O}) = 0$ $\Rightarrow$ Question B YES

Question C

Given

- $z_1, \dots z_n \in X$, $p_1 \dots p_m \in X$
- $\mu_1, \dots \mu_n \geq 0$, $\nu_1, \dots \nu_m \geq 0$ integers

Want f meromorphic in X

- f has zeroes at z_i of order $\geq \mu_i$
- f has poles at p_i of order $\leq \nu_i$

Other zeroes are allowed, but no other poles.

Let $D = -\sum_i \mu_i [z_i] + \sum_i \nu_i [p_i]$

Want $\operatorname{div} f - \sum_i \mu_i [z_i] + \sum_i \nu_i [p_i] \geq 0$

$\iff$ $D + \operatorname{div} f \geq 0$ (non-negative coefficients).

Sheaves associated to divisors

Given a divisor D, we form a sheaf $\mathcal{O}_X(D)$.

via the assignment

$$U \longrightarrow \{ f \text{ meromorphic} \neq 0,\ \operatorname{div} f + D|_U \geq 0 \} \cup \{0\}.$$

if U connected.

Conclusion Question C is asking to describe

$$V_D = H^0(X, \mathcal{O}_X(D)).$$

Gustav Roch (1839 – 1866)

While in Göttingen, Roch attended lectures of Riemann.

Example $X = \mathbb{C}/\Lambda$, $D = d[0]$

$$V_d = \{ f \text{ elliptic}, f \text{ has pole at } 0 \text{ of order } \leq d \}$$

Claim $\dim V_d = d$.

Proof Recall the Weierstraß $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \dots$$

$\Rightarrow$ $\wp$ has pole of order 2 at 0

$\Rightarrow$ $\wp'$ has pole of order 3 at 0

$\Rightarrow$ $\wp^{(k)}$ has pole of order $k+2$ at 0.

$\Rightarrow$ $1, \wp, \wp', \dots, \wp^{(d-2)}$ are in V_d.

Claim $1, \wp, \wp', \dots, \wp^{(d-2)}$ are independent.

Proof If

$$\underbrace{\wp^{(k)}}_{\text{order } k+2} = \underbrace{\sum_{j=0}^{k-1} a_j \wp^{(j)}}_{\text{order } \leq j+2 \leq k+1}$$

we inspect the order of the pole at 0.

Contradiction!

Conclusion $\dim V_d \geq d$.

We will show the opposite inequality.

Claim: $\dim V_d \leq d$.

Let f be an elliptic function in V_d. Laurent expand

$$f = \frac{a_{-d}}{z^d} + \cdots + \frac{a_{-1}}{z} + a_0 + \cdots$$

Note $a_{-1} = \frac{1}{2\pi i} \int_{\partial P} f \, dz = 0$. by the Residue Theorem

& periodicity of f.

The coefficients $(a_{-d}, \ldots, a_{-2}, a_0)$ determine f

at most uniquely $\Rightarrow \dim V_d \leq d$.

Indeed, assume f_1, f_2 are elliptic and have the same Laurent coefficients a_{-k}. Then

$f_1 - f_2$ = holomorphic at 0 & everywhere else

(by definition of V_d) & Λ-periodic

= constant by Liouville. & vanishing at 0

$\Rightarrow f_1 - f_2 \equiv 0 \Rightarrow f_1 \equiv f_2$.

Sheaves on Riemann Surfaces

- $\mathcal{O}$ = sheaf of holomorphic functions
- $\mathcal{O}^*$ = sheaf of holomorphic nonvanishing fns.
- $\mathcal{M}$ = sheaf of meromorphic functions
- $\mathcal{M}^*$ = nonzero meromorphic functions
- $\underline{\mathbb{C}}$ = sheaf of locally constant functions
- $\mathcal{C}^\infty$ = sheaf of smooth functions
- $\underline{\mathcal{D}iv}$ = sheaf of divisors
- $\mathcal{O}_X(D)$ = sheaf associated to divisor D.

[I] These sheaves solve different problems

[II] These sheaves interact with each other