Math 220C - Lecture 24

May 28, 2021

§ 1. Homological Methods & Sheaves (Part II)

Given $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves, we define

$\boxed{i}$ the sheaf $\operatorname{Ker} \alpha$

$\boxed{ii}$ the sheaf $\operatorname{Coker} \alpha$

Kernel When $U \subseteq X$ open, set

$$\operatorname{Ker} \alpha \, (U) = \operatorname{Ker} \{ \alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \}$$

Restriction maps

$\operatorname{Ker} \alpha \, (U) \longrightarrow \operatorname{Ker} \alpha \, (V)$ are naturally defined.

Check $\boxed{i}$ $\operatorname{Ker} \alpha$ is a sheaf

$\boxed{ii}$ $(\operatorname{Ker} \alpha)_x = \operatorname{Ker} \{ \alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \}.$

Cokernel Presheaf

For $U \subseteq X$ open, define

$$\widetilde{\operatorname{Coker} \alpha}\,(U) = \operatorname{Coker} \{ \alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \}$$

Check

- [ ] $\widetilde{\operatorname{Coker} \alpha}$ is a presheaf
- [ ] $\left(\widetilde{\operatorname{Coker} \alpha} \right)_x = \operatorname{Coker} (\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x)$

Beware! $\widetilde{\operatorname{Coker} \alpha}$ is not always a sheaf

Example $X = \mathbb{C}^\times$, $\mathcal{O} \xrightarrow{\alpha} \mathcal{O}^*$, $\alpha(f) = e^{2\pi i f}$

Let $\mathcal{F} = \widetilde{\mathrm{Coker}\,\alpha}$

- $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $\mathcal{F}(U) = 0$

Indeed, U is simply connected, so logarithms make sense $\Rightarrow$ α is surjective in U $\Rightarrow$ $\widetilde{\mathrm{Coker}}\,\alpha_U = 0$.

- $V = \mathbb{C} \setminus \mathbb{R}_{\geq 0} \Rightarrow \mathcal{F}(V) = 0$

- $X = U \cup V = \mathbb{C}^\times$. Note α is not surjective in X since $\log z$ is not defined in $\mathbb{C}^\times$. $\Rightarrow \mathcal{F}(X) \neq 0$.

Any $s \in \mathcal{F}(X)$ restricts $s|_U = 0$, $s|_V = 0$ since $\mathcal{F}(U) = 0$, $\mathcal{F}(V) = 0$. If $s \neq 0$ this contradicts uniqueness of gluing $\Rightarrow$ $\mathcal{F}$ not a sheaf.

Sheafification

Goal

Given $\mathcal{F} \longrightarrow X$ a presheaf, we define a sheaf $\mathcal{F}^{\#}$

& morphism of presheaves

$$i : \mathcal{F} \longrightarrow \mathcal{F}^{\#}.$$

Remark In addition,

[i] $\mathcal{F}$ sheaf $\Rightarrow$ $\mathcal{F} = \mathcal{F}^{\#}$

[ii] $\mathcal{F}_x = \mathcal{F}^{\#}_x \quad \forall x$

[iii] Given $\mathcal{F} \longrightarrow \mathcal{G}$ we obtain $\mathcal{F}^{\#} \longrightarrow \mathcal{G}^{\#}$ with

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F}^{\#} \arrow[r] & \mathcal{G}^{\#} \end{tikzcd}$$

commutative.

Definition

$$\mathcal{F}^{\#}(U) = \Big\{ (f_x)_{x\in U} \in \mathcal{F}_x, \text{ "locally compatible germs" i.e.}$$

$$\forall\, x \in U \quad \exists\, x \in V \subseteq U,\ s \in \mathcal{F}(V) \text{ with } s_y = f_y \ \ \forall\, y \in V \Big\}$$

Example

$\mathcal{F}$ = presheaf of constant functions

$\mathcal{F}^{\#}$ = sheaf of locally constant functions

Remark We define $\mathcal{F} \longrightarrow \mathcal{F}^{\#}$ via

$$\mathcal{F}(U) \ni f \longrightarrow (f_x)_x \in \mathcal{F}^{\#}(U).$$

Check $\mathcal{F}^{\#}$ is a sheaf & $\mathcal{F}_x = \mathcal{F}^{\#}_x$.

Conclusion — Cokernel sheaf

Given $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$, we define the cokernel sheaf:

(1) $\widetilde{\operatorname{Coker} \alpha}$ presheaf

(2) sheafify $\operatorname{Coker} \alpha := \widetilde{\operatorname{Coker} \alpha}^{\#}$.

Why does it work?

Assume $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$. We have by definition

$$\mathcal{G} \longrightarrow \widetilde{\operatorname{Coker} \alpha}$$

This gives

$$\mathcal{G} = \mathcal{G}^{\#} \longrightarrow \widetilde{\operatorname{Coker} \alpha}^{\#}.$$

Then $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \widetilde{\operatorname{Coker} \alpha}^{\#} \longrightarrow 0$ exact

as needed.

Exactness can be checked on stalks. We note that

$$\left(\widetilde{\operatorname{Coker}\alpha}^{\#}\right)_x = \left(\widetilde{\operatorname{Coker}\alpha}\right)_x$$

$$= \operatorname{Coker}(\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x).$$

$$\Rightarrow \quad 0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \left(\widetilde{\operatorname{Coker}\alpha}^{\#}\right)_x \longrightarrow 0$$

exact, as needed.

§ 2. Flabby sheaves

$\mathcal{F}$ is **flabby** provided $\forall\ V \subseteq U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \text{ is surjective.}$$

Remarks $\boxed{\text{i}}$ $\mathcal{F}$ flabby $\Rightarrow$ $\mathcal{F}|_U$ flabby $\forall\ U \subseteq X$ open

Indeed, for $V \subseteq W \subseteq U$,

$$\mathcal{F}|_U(W) = \mathcal{F}(W) \longrightarrow \mathcal{F}(V) = \mathcal{F}|_U(V) \quad \text{surjective.}$$

$\boxed{\text{ii}}$ Suffices to check $\forall\ U \subseteq X$ open

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \quad \text{surjective}$$

Indeed,
$$\begin{tikzcd} \mathcal{F}(X) \arrow[rr] \arrow[dr] & & \mathcal{F}(U) \\ & \mathcal{F}(V) \arrow[ur] & \end{tikzcd}$$
shows $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ also surjective for $U \subseteq V$.

Key Lemma

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact, $\mathcal{F}$ flabby then

$\boxed{\text{i}}$ $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact, $\forall\, U \subseteq X$ open

$\boxed{\text{ii}}$ $\mathcal{F}, \mathcal{G}$ flabby $\Rightarrow \mathcal{H}$ flabby

Proof $\boxed{\text{i}} \Rightarrow \boxed{\text{ii}}$

Let $V \subseteq U$ open. Compare the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, "\text{surj}"] & \mathcal{G}(U) \arrow[r] \arrow[d, "\text{surj}"] \arrow[r, phantom, shift right=6ex, "\Rightarrow"] & \mathcal{H}(U) \arrow[r] \arrow[d, "\text{surj}"] & 0 \\
0 \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \arrow[r] & \mathcal{H}(V) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \mathcal{H}$ flabby.

Proof of II

WLOG $\mathcal{U} = X$ else work the restrictions:

$$0 \longrightarrow \mathcal{F}/_{u} \longrightarrow \mathcal{G}/_{u} \longrightarrow \mathcal{H}/_{u} \longrightarrow 0$$ exact & $\mathcal{F}/_{u}$ flabby

Suffices

$\beta : \mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ surjective if $\mathcal{F}$ flabby

Proof Let $h \in \mathcal{H}(X)$. Define

$$\mathcal{A} = \{ (g, u) : g \in \mathcal{G}(u) \text{ and } \beta(g) = h/_{u} \}.$$

Define

$(g, u) \geq (g', u')$ if $u \supseteq u'$ & $g/_{u'} = g'$

Remark Every linearly ordered chain admits an upper bound (take the union).

$\overset{\text{Zorn}}{\Longrightarrow}$ $\mathcal{A}$ admits a maximal element $(g, \mathcal{U})$.

Claim $\mathcal{U} = X$ for the maximal $(g, \mathcal{U})$.

This gives $\beta : \mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ surjective.

Proof of the claim

(1) $\mathcal{U} \neq X$. We obtain a contradiction. Let $p \in X \setminus \mathcal{U}$.

(2) $\beta_p : \mathcal{G}_p \longrightarrow \mathcal{H}_p$ surjective $\Rightarrow$

$\Rightarrow \exists\ \tilde{g}_p$ with $\beta_p(\tilde{g}_p) = h_p$

$\Rightarrow \exists\ V\ \exists\ \tilde{g} \in \mathcal{G}(V),\ \beta(\tilde{g}) = h/_V$.

(3) Overlaps: $U \cap V$

$$\beta(\tilde{g}|_{U\cap V}) = \beta(g|_{U\cap V}) = h|_{U\cap V}$$

$$\Rightarrow \boxed{\tilde{g}|_{U\cap V} - g|_{U\cap V} = \alpha(f)}, \quad f \in \mathcal{F}(U\cap V).$$

This uses $0 \longrightarrow \mathcal{F}(U\cap V) \longrightarrow \mathcal{G}(U\cap V) \longrightarrow \mathcal{H}(U\cap V)$

exact, as proved in Lecture 23.

(4) $\mathcal{F}$ flabby $\Rightarrow$ extend f to X.

(5) Define $W = U \cup V$ and

$$\tilde{\tilde{g}} = \begin{cases} g & \text{in } U \\ \tilde{g} - \alpha(f) & \text{in } V \end{cases}$$ well-defined

Note $\beta(\tilde{\tilde{g}}) = h \Rightarrow (\tilde{\tilde{g}}, W) \in \mathcal{A}$.

This element contradicts maximality of (g, U).