Math 220C - Lecture 25

June 2, 2021

Last time

$\mathcal{F} \longrightarrow X$ flabby if $\forall\, V \subseteq U \subseteq X$ open,

$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ surjective

Key Lemma

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact, $\mathcal{F}$ flabby then

$\boxed{i}$ $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact, $\forall\, U \subseteq X$ open

$\boxed{ii}$ $\mathcal{F}, \mathcal{G}$ flabby $\Rightarrow$ $\mathcal{H}$ flabby

§1. Main Theorem of Sheaf Cohomology

$\exists$ functors

$$H^p(X, -) : \text{Sheaves on } X \longrightarrow \text{Abelian Groups}$$

such that

$\boxed{a}$ $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

$\boxed{b}$ $\mathcal{F}$ flabby $\Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1$

$\boxed{c}$ Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

$\exists$ connecting homomorphisms

$$\delta_p : H^p(X, \mathcal{H}) \longrightarrow H^{p+1}(X, \mathcal{F})$$

functorial in exact sequences such that

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \xrightarrow{\delta_0} \\ \hookrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \xrightarrow{\delta_1} \\ \hookrightarrow \cdots \end{aligned}$$

exact.

These requirements determine the functors uniquely.

Aside from Homological Algebra

(1) Given a complex $\quad d \circ d = 0$

$$\to A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \to A^n \to \cdots$$

we define $\quad H^p(A^\bullet) = \dfrac{\mathrm{Ker}\; A^p \xrightarrow{d} A^{p+1}}{\mathrm{Im}\; A^{p-1} \xrightarrow{d} A^p}.$

(2) Given complexes $A^\bullet, B^\bullet, C^\bullet$ such that

$$0 \to A^p \to B^p \to C^p \to 0 \quad \text{exact} \quad \forall p$$

we write $\quad 0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0. \quad$ exact

(3) If $\quad 0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0 \quad$ exact $\quad$ then

$$\hookrightarrow H^p(A^\bullet) \to H^p(B^\bullet) \to H^p(C^\bullet)$$

$$\hookrightarrow H^{p+1}(A^\bullet) \to H^{p+1}(B^\bullet) \to H^{p+1}(C^\bullet)$$

$$\hookrightarrow \cdots$$

exact.

Outline of the argument

[i] every sheaf $\mathcal{F}$ admits a canonical flabby resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \quad , \quad \mathcal{F}^p \text{ flabby}$$

[ii] Take global sections. We obtain the complex

$$\mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \mathcal{F}^2(X) \longrightarrow \cdots$$

[iii] Define

$$H^p(X, \mathcal{F}) = \frac{\operatorname{Ker} \quad \mathcal{F}^p(X) \longrightarrow \mathcal{F}^{p+1}(X)}{\operatorname{Im} \quad \mathcal{F}^{p-1}(X) \longrightarrow \mathcal{F}^p(X)} .$$

[iv] Show this works.

§2. Preparation for the proof — The Godement Sheaf

Definition Given $\mathcal{F} \to X$, define the sheaf $\Phi\mathcal{F}$ via

$$\Phi\mathcal{F}(U) = \prod_{x \in U} \mathcal{F}_x .$$

This is called the sheaf of totally discontinuous sections.

Remarks

[1] We define $\mathcal{F} \to \phi\mathcal{F}$ sending

$$f \longmapsto (f_x)_{x \in U} \quad \text{for } f \in \mathcal{F}(U).$$

We have $\mathcal{F} \to \phi\mathcal{F}$ injective. Indeed,

$$f = 0 \iff f_x = 0 \quad \forall x \in U,$$ see Lecture 23

$\boxed{ii}$ $\Phi\mathcal{F}$ is flabby. Indeed, for $U \supseteq V$,

$$\phi\,\mathcal{F}(U) \longrightarrow \phi\,\mathcal{F}(V) \quad \text{surjective}$$

$$\iff \quad \prod_{x\in U} \mathcal{F}_x \longrightarrow \prod_{x\in V} \mathcal{F}_x \quad \text{surjective, which is clear}$$

$\boxed{iii}$ If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact then

$$\Downarrow$$

$$0 \longrightarrow \phi\,\mathcal{F} \longrightarrow \phi\,\mathcal{G} \longrightarrow \phi\,\mathcal{H} \longrightarrow 0 \quad \text{exact}$$

Why? For all $U \subseteq X$ open, we have

$$0 \longrightarrow \phi\,\mathcal{F}(U) \longrightarrow \phi\,\mathcal{G}(U) \longrightarrow \phi\,\mathcal{H}(U) \longrightarrow 0 \quad \text{exact.}$$

Indeed, this is because

$$0 \longrightarrow \prod_{x\in U} \mathcal{F}_x \longrightarrow \prod_{x\in U} \mathcal{G}_x \longrightarrow \prod_{x\in U} \mathcal{H}_x \longrightarrow 0 \quad \text{exact}$$

which follows since $0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$ exact.

IV Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$$
\begin{tikzcd}
 & \text{exact} & \text{exact} & \text{exact} & & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 & \text{exact} \\
0 \arrow[r] & \phi\,\mathcal{F} \arrow[r] \arrow[d] & \phi\,\mathcal{G} \arrow[r] \arrow[d] & \phi\,\mathcal{H} \arrow[r] \arrow[d] & 0 & \text{exact by } \boxed{III} \\
0 \arrow[r] & \phi\,\mathcal{F}/\mathcal{F} \arrow[r] \arrow[d] & \phi\,\mathcal{G}/\mathcal{G} \arrow[r] \arrow[d] & \phi\,\mathcal{H}/\mathcal{H} \arrow[r] \arrow[d] & 0 & \Downarrow \ \text{exact} \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

The last row is exact. This can be checked on stalks.

In the diagram of stalks, the columns & first 2 rows are exact $\Longrightarrow$ 3rd row is also exact.

The canonical flabby resolution

Given $\mathcal{F} \longrightarrow X$ we construct a resolution

$$(*) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots$$

where $\mathcal{F}^p$ are flabby $\forall\, p \geq 0$.

How? Form the exact sequences:

$$(0) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \phi\,\mathcal{F} \longrightarrow \tilde{\mathcal{F}}^1 = \phi\,\mathcal{F}/\mathcal{F} \longrightarrow 0$$

$$(1) \qquad 0 \longrightarrow \tilde{\mathcal{F}}^1 \longrightarrow \phi\,\tilde{\mathcal{F}}^1 \longrightarrow \tilde{\mathcal{F}}^2 = \phi\,\tilde{\mathcal{F}}^1/\tilde{\mathcal{F}}^1 \longrightarrow 0$$

$\vdots$

$$(p) \qquad 0 \longrightarrow \tilde{\mathcal{F}}^p \longrightarrow \phi\,\tilde{\mathcal{F}}^p \longrightarrow \tilde{\mathcal{F}}^{p+1} = \phi\,\tilde{\mathcal{F}}^p/\tilde{\mathcal{F}}^p \longrightarrow 0$$

where we define

- $\tilde{\mathcal{F}}^0 = \mathcal{F}$
- $\mathcal{F}^p = \phi\,\tilde{\mathcal{F}}^p$ = flabby
- $\tilde{\mathcal{F}}^{p+1} = \phi\,\tilde{\mathcal{F}}_p/\tilde{\mathcal{F}}_p$

The resolution (*) follows by concatenating the above exact sequences.

Functoriality of the Godement resolution

Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

e.g. $0 \longrightarrow \widetilde{\mathcal{F}}^0 \longrightarrow \widetilde{\mathcal{G}}^0 \longrightarrow \widetilde{\mathcal{H}}^0 \longrightarrow 0$ exact.

We show $0 \longrightarrow \widetilde{\mathcal{F}}^p \longrightarrow \widetilde{\mathcal{G}}^p \longrightarrow \widetilde{\mathcal{H}}^p \longrightarrow 0$ exact $\forall p$.

We use induction on p. The case $p=0$ is clear.

For the inductive step, we use $\boxed{\text{IV}}$ & diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{\mathcal{F}}^p \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^p \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}^p \arrow[r] \arrow[d] & \mathcal{G}^p \arrow[r] \arrow[d] & \mathcal{H}^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\mathcal{F}}^{p+1} \arrow[r] \arrow[d] & \tilde{\mathcal{G}}^{p+1} \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^{p+1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

exact

exact by $\boxed{III}$

$\Downarrow$

exact

This also shows $0 \longrightarrow \mathcal{F}^p \longrightarrow \mathcal{G}^p \longrightarrow \mathcal{H}^p \longrightarrow 0$ exact.

Conclusion

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$$\Downarrow$$

$0 \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet \longrightarrow 0$ exact.

$$\Downarrow \quad \text{Key Lemma}$$

$0 \longrightarrow \mathcal{F}^\bullet(X) \longrightarrow \mathcal{G}^\bullet(X) \longrightarrow \mathcal{H}^\bullet(X) \longrightarrow 0$ exact

Proof of the theorem

[i] Godement resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \quad , \quad \mathcal{F}^p \text{ flabby}$$

[ii] Global sections

$$\mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \mathcal{F}^2(X) \longrightarrow \cdots$$

[iii] Define

$$H^p(X, \mathcal{F}) = \frac{\operatorname{Ker} \quad \mathcal{F}^p(X) \longrightarrow \mathcal{F}^{p+1}(X)}{\operatorname{Im} \quad \mathcal{F}^{p-1}(X) \longrightarrow \mathcal{F}^p(X)} .$$

We verify it works!

Property $\boxed{a}$

WTS $\quad H^0(X, \mathcal{F}) = \operatorname{Ker} \mathcal{F}^0(X) \xrightarrow{\gamma\beta} \mathcal{F}^1(X) \overset{?}{=} \mathcal{F}(X).$

Recall the sequences

$$(0) \qquad 0 \longrightarrow \mathcal{F}(X) \xrightarrow{\alpha} \mathcal{F}^0(X) \xrightarrow{\beta} \tilde{\mathcal{F}}^1(X)$$

$$(1) \qquad 0 \longrightarrow \tilde{\mathcal{F}}^1(X) \xrightarrow{\gamma} \mathcal{F}^1(X) \longrightarrow \tilde{\mathcal{F}}^2(X)$$

This shows:

γ injective

$$\operatorname{Ker} \gamma\beta = \operatorname{Ker} \beta \quad \text{by } (1)$$

$$= \operatorname{Im} \alpha \cong \mathcal{F}(X) \quad \text{by } (0).$$

Property [6]

WTS : $\mathcal{F}$ flabby $\Rightarrow H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$

- $\mathcal{F}^p = \phi\ \tilde{\mathcal{F}}^p$ flabby $\forall p \geq 0$ by [1] above
- $\tilde{\mathcal{F}}^p$ flabby $\forall p \geq 0$, $\tilde{\mathcal{F}}^0 = \mathcal{F}$ = flabby (given)

Why? Induct on p. & use the sequence.

$$0 \longrightarrow \tilde{\mathcal{F}}^p \longrightarrow \mathcal{F}^p \longrightarrow \tilde{\mathcal{F}}^{p+1} \longrightarrow 0 \qquad \text{exact}$$

The Key Lemma shows $\tilde{\mathcal{F}}^p$ flabby $\Rightarrow \tilde{\mathcal{F}}^{p+1}$ flabby.

Also by the Key Lemma, we have

- $0 \longrightarrow \tilde{\mathcal{F}}^p(X) \longrightarrow \mathcal{F}^p(X) \longrightarrow \tilde{\mathcal{F}}^{p+1}(X) \longrightarrow 0$ exact

$\Downarrow$

$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}^0(X) \longrightarrow \mathcal{F}^1(X) \longrightarrow \cdots$ exact

$\Longrightarrow$ no cohomology $H^p(X, \mathcal{F}) = 0 \quad \forall p \geq 1.$

Property $\boxed{c}$

Assume $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact

$$\Downarrow$$

$0 \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet \longrightarrow 0$ exact

$$\Downarrow$$

$0 \longrightarrow \mathcal{F}^\bullet(X) \longrightarrow \mathcal{G}^\bullet(X) \longrightarrow \mathcal{H}^\bullet(X) \longrightarrow 0$ exact

$$\Rightarrow \quad \hookrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{G}) \longrightarrow H^p(X, \mathcal{H})$$

$$\hookrightarrow H^{p+1}(X, \mathcal{F}) \longrightarrow \cdots$$

using the facts from homological algebra reviewed above for

$$A^\bullet = \mathcal{F}^\bullet(X), \quad B^\bullet = \mathcal{G}^\bullet(X), \quad C^\bullet = \mathcal{H}^\bullet(X).$$