Math 220C - Lecture 3

April 2, 2021

Last time $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ.

$\boxed{I}$ $$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{is})\, ds$$ Mean Value Property

$\boxed{II}$ $$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\Phi(e^{is}))\, ds$$ MVP + Aut Δ to recenter.

where $\Phi : \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$, $z \longrightarrow \dfrac{z+a}{1+\bar{a}z}$

with inverse $\varphi : \Delta \longrightarrow \Delta$, $\partial\Delta \longrightarrow \partial\Delta$, $z \longrightarrow \dfrac{z-a}{1-\bar{a}z}$

Goal Make formula $\boxed{II}$ even more explicit.

Poisson Kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \cos n\theta, \quad 0 \le r < 1 \quad \text{well defined.}$$

Three additional Formulas

$\boxed{a}$ $$P_r(\theta) = \operatorname{Re} \frac{1+z}{1-z}, \quad z = re^{i\theta}.$$

$$\frac{1+z}{1-z} = 1 + \frac{2z}{1-z} = 1 + 2z(1+z+z^2+\dots)$$

$$= 1 + 2\sum_{n=1}^{\infty} z^n = 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n (\cos n\theta + i \sin n\theta).$$

$$\operatorname{Re} \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta$$

$$= 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta}).$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. = P_r(\theta)$$

$\boxed{b}$ $$P_r(\theta) = \frac{1-|z|^2}{|1-z|^2}.$$

$\boxed{a}$

$$P_r(\theta) = \mathrm{Re}\,\frac{1+z}{1-z} = \mathrm{Re}\,\frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})}$$

$$= \mathrm{Re}\,\frac{1-z\bar{z}+\overbrace{z-\bar{z}}^{\text{imaginary}}}{|1-z|^2}$$

$$= \frac{1-|z|^2}{|1-z|^2}.$$

$\boxed{c}$ $$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$ very useful.

Indeed use $\boxed{b}$ for $z = re^{i\theta}$:

$$|1-z|^2 = (1-r\cos\theta)^2 + (r\sin\theta)^2$$

$$= 1+r^2-2r\cos\theta \quad \& \quad 1-|z|^2 = 1-r^2.$$

Poisson's Formula

$u : \bar{\Delta} \to \mathbb{R}$ continuous & harmonic in Δ, $a = re^{i\theta}$

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, dt.$$

Proof Recall

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\Phi(e^{is}))\, ds$$

Change of variables $e^{is} = \psi(e^{it})$

Main Claim $ds = P_r(\theta - t)\, dt$

the Poisson kernel arises via change of variables

Assuming this, we obtain

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(\Phi\, \psi(e^{it})) . \ P_r(\theta - t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\, P_r(\theta - t)\, dt. \text{, as needed.}$$

Proof of the Main claim

Chain rule

$$ds = \frac{d(e^{is})}{i e^{is}} = \frac{d\,\varphi(e^{it})}{i\,\varphi(e^{it})} = \frac{\varphi'(e^{it})\; i\; e^{it}\,dt}{i\;\varphi(e^{it})} = \frac{\varphi'(z)\,z}{\varphi(z)}\,dt$$

Recall $\varphi(z) = \dfrac{z-a}{1-\bar{a}z}$. Taking logarithmic derivatives

$$z \cdot \frac{\varphi'(z)}{\varphi(z)} = \frac{z}{z-a} + \frac{\bar{a}z}{1-\bar{a}z}$$

$$= \frac{z}{z-a} - \frac{1}{2} + \frac{1}{2} + \frac{\bar{a}z}{1-\bar{a}z}$$

$$= \frac{1}{2}\cdot\frac{z+a}{z-a} + \frac{1}{2}\cdot\frac{1+\bar{a}z}{1-\bar{a}z}\,.$$

$1 = z\bar{z}$

$$= \frac{1}{2}\cdot\frac{z+a}{z-a} + \frac{1}{2}\cdot\frac{z\bar{z}+\bar{a}z}{z\bar{z}-\bar{a}z}$$

$$= \frac{1}{2}\cdot\frac{z+a}{z-a} + \frac{1}{2}\cdot\frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}$$

$$= \operatorname{Re}\frac{z+a}{z-a} = \operatorname{Re}\frac{1+\frac{a}{z}}{1-\frac{a}{z}} = P_r(\theta-t)$$

using $\boxed{a}$ & $\dfrac{a}{z} = \dfrac{re^{i\theta}}{e^{it}} = r\,e^{i(\theta-t)}$

Siméon Poisson

(1781 – 1840)

Students:

Liouville, Carnot, Dirichlet

Poisson Kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

$$= \operatorname{Re} \frac{1+z}{1-z} = \frac{1-|z|^2}{|1-z|^2}, \quad \text{for } z = re^{i\theta}.$$

Poisson integral formula

$u : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ. Then

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, dt$$

Remark We can dilate & translate to work with any disc $\Delta(a, R)$.

Theorem $u : \overline{\Delta}(a,R) \longrightarrow \mathbb{R}$ continuous & harmonic in $\Delta(a,R)$.

$$u(a + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2}\, u(a + Re^{it})\, dt$$

Proof

$$\tilde{u} : \overline{\Delta} \longrightarrow \mathbb{R}, \quad \tilde{u}(z) = u(a + Rz)$$

We apply the previous result to $\tilde{u}$. Then

$$u(a + re^{i\theta}) = \tilde{u}\left(\frac{r}{R}e^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} P_{\frac{r}{R}}(\theta - t)\, \tilde{u}(e^{it})\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - \left(\frac{r}{R}\right)^2}{1 - 2\frac{r}{R}\cos(\theta - t) + \left(\frac{r}{R}\right)^2}\, u(a + Re^{it})\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2}\, u(a + Re^{it})\, dt.$$

Two Consequences

[i] Schwarz Integral Formula

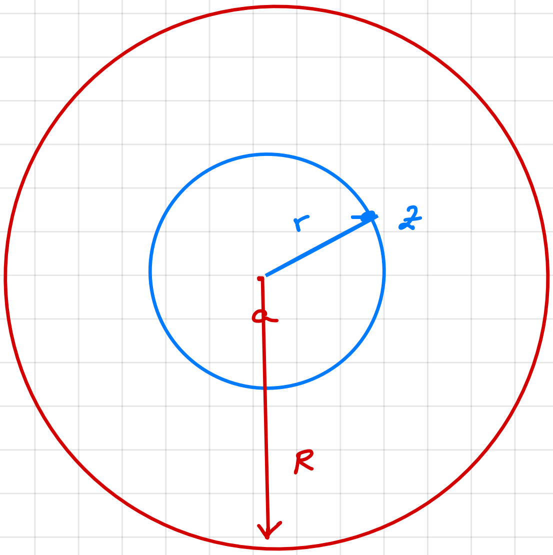
[ii] Harnack Inequality

Harnack's Inequality

$u : \bar{\Delta}(a, R) \longrightarrow \mathbb{R}$ continuous, harmonic in $\Delta(a, R)$. & $u \geq 0$.

If $|z - a| = r \implies$

$$u(a) \cdot \frac{R - r}{R + r} \leq u(z) \leq u(a) \frac{R + r}{R - r}$$



Proof

We use $-1 \leq \cos(\theta - t) \leq 1$.

The two inequalities are similar. For instance, 2^{nd} inequality:

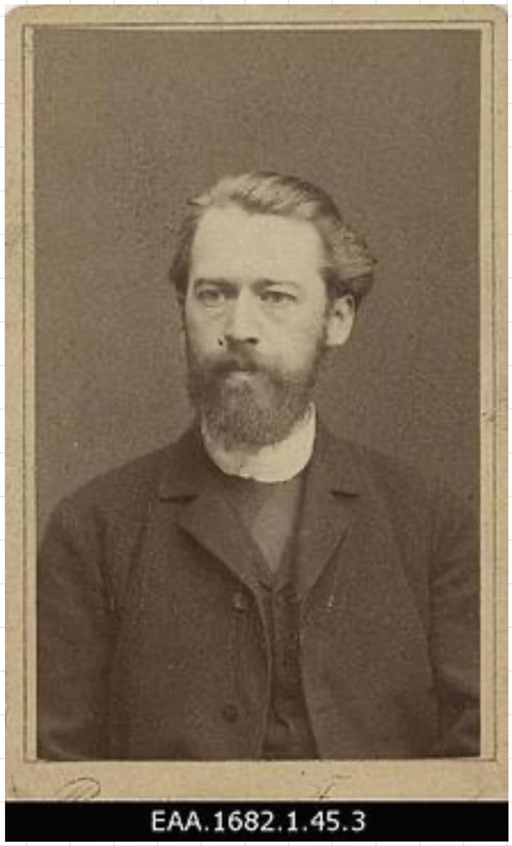
$$u(a + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2}\, u(a + Re^{it})\, dt$$

$u \geq 0$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr + r^2} \cdot u(a + Re^{it})\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cancel{(R-r)}(R+r)}{(R-r)^{\cancel{2}}} \cdot u(a + Re^{it})\, dt$$

$$= u(a)\frac{R+r}{R-r}$$ using Mean Value Property.



Axel Harnack (1851-1888) was a Baltic-German mathematician.

He proved Harnack's inequality for harmonic functions & Harnack's curve theorem in real algebraic geometry.