\[
\text{Last time:} \quad u : \overline{\Delta} \to \mathbb{R} \quad \text{continuous, harmonic in } \Delta
\]

\[
u(a) = \frac{1}{2\pi} \int_{0}^{2\pi} P_{r}(\theta - t) \, u(\epsilon + i \theta) \, dt \quad (\text{Poisson})
\]

\[
a = r e^{i\theta}
\]

\[
\text{Poisson Kernel} \quad P_{r}(\theta) = \text{Re} \left( \frac{1 + r e^{i\theta}}{1 - r e^{i\theta}} \right)
\]

\[
= \frac{1 - r^2}{1 - 2r \cos \theta + r^2}
\]

\[
\text{Applications} \quad \text{Harnack Inequality} \quad \text{Schwarz Integral Formula}
\]
Schwarz Integral Formula

\[ u : \Delta \rightarrow \mathbb{R} \text{ continuous, harmonic in } \Delta \]

We have seen \( u = \text{Re} f \), \( f \) holomorphic in \( \Delta \).

**Question**

Is there a formula for \( f \)?

\[ f : \Delta \rightarrow \mathbb{C} , \quad f(a) = \frac{1}{2 \pi i} \int_{\partial \Delta} \frac{2 + a}{2 - a} u(z) \frac{dz}{z} \]

**Claims**

(i) \( f \) holomorphic in \( \Delta \).

(ii) \( u = \text{Re} f \)
Proof of (1)

**Key Fact** (Math 220A, Homework 3, Problem 7).

Continuous \( \Phi: \{ \gamma \} \times \mathbb{U} \to \mathbb{D} \) holomorphic in \( \mathbb{D} \)

then \( \Phi \) holomorphic

Apply this to \( \Phi: \mathbb{D} \times \mathbb{D} \to \mathbb{D}, \Phi(z, a) = \frac{z + a}{z - a} \frac{u(a)}{\overline{a}} \).

which is continuous and holomorphic in \( \mathbb{D} \) to conclude.

\[
\mathcal{F}(a) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z + a}{z - a} u(z) \frac{dz}{z} \text{ is holomorphic in } \mathbb{D}.
\]
**Proof of (2)**

By definition, we have

\[ f(a) = \frac{1}{2\pi} \int_{|z| = 1} \frac{2 + a}{2 - a} \frac{I(z)}{z} \, dz \]

\[ = \frac{1}{2\pi} \int_{|z| = 1} \frac{1 + a^2}{1 - a^2} \frac{u(e^{it})}{i \, dt} \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} \text{Re} \left( \frac{1 + a^2}{1 - a^2} \right) u(e^{it}) \, dt \]

\[ = \text{Re} \left( \frac{1 + a^2}{1 - a^2} \right) u(a) \]

\[ \Rightarrow u(a) = \text{Re} f(a) \]

In the last line we applied Poisson's formula for \( u \).
Hermann Schwarz
1843 - 1921

Doctoral advisor:
Karl Weierstrass
Ernst Kummer

Students:
Lipót Fejér
Paul Koebe
Ernst Zermelo

Schwarz lemma
Schwarz integral formula
Schwarz reflection principle
Cauchy–Schwarz inequality.
Dirichlet Problem (for the unit disc)

Given \( f: \Delta \to \mathbb{R} \) continuous, is there \( u: \overline{\Delta} \to \mathbb{R} \) continuous

\begin{align*}
(1) & \quad u \text{ harmonic in } \Delta \\
(2) & \quad u / \partial \Delta = f
\end{align*}

Answer Yes. Define \( u: \overline{\Delta} \to \mathbb{R} \) by

\[
\frac{1}{2\pi} \int_0^{2\pi} L_r (\theta - t) f(e^{it}) \, dt, \quad r < 1.
\]

We need to show

\begin{align*}
(1) & \quad u \text{ harmonic in } \Delta \\
(2) & \quad u \text{ continuous in } \overline{\Delta}
\end{align*}

\[
\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} L_r (\theta - t) f(e^{it}) \, dt = f(e^{i\phi_0}).
\]
Johann Peter Gustav Lejeune Dirichlet (1805 – 1859)

It was his father who first went under the name “Lejeune Dirichlet” (meaning “the young Dirichlet”) in order to differentiate from his father, who had the same first name. The name “Dirichlet” (or “Derichelette”) means “from Richelette” after a town in Belgium.
Proof of (i)

We claim that \( u \) is harmonic in \( \Delta \). Recall that

\[
u(a) = \frac{1}{2\pi i} \int_0^{2\pi} P_r(a - t) f(r^i) \, dt, \quad a \in \Delta.
\]

Let

\[
g(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{2 + a}{z - a} \cdot f(z) \, \frac{dz}{z}.
\]

We have argued in the proof of Schwarz, \( g \) is holomorphic on \( \Delta \) and \( \Re g = u \). Thus \( u \) is harmonic in \( \Delta \).
Proof of (2)

Properties of the Poisson kernel

Lemma

1. \( P_r(t) \geq 0 \), even in \( t \), \( 2\pi \)-periodic in \( t \).

2. \( \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \, dt = 1 \).

3. \( P_r \to 0 \) as \( r \to 1 \), over the domain \( \delta \leq |t| \leq \pi \) and \( \delta > 0 \).

Proof

1. is clear.

2. Take \( u \equiv 1 \), \( a = r \) i.e. in Poisson's formula

\[
1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t) \, dt,
\]

which is what we need.

3. To prove uniform convergence, we show

\[
\sup_{\delta \leq t \leq \pi} |P_r(t)| \to 0 \text{ as } r \to 1.
\]

Note that \( P_r \) is decreasing in \( t \in [\delta, \pi] \). Then

\[
\sup_{\delta \leq t \leq \pi} P_r(t) = P_r(\delta) = \frac{1-r^2}{1-2r \cos \delta + r^2} \to 0 \text{ as } r \to 1.
\]
Heuristics

- Area under the graph: is 1. by [π]

- End points: \( P_r(t) \to 0 \) as \( r \to 1 \)
  for \( t \in [\delta, 1] \).

- Most area concentrated in the middle
  \[ P_r(0) = \frac{1+r}{1-r} \to \infty. \]

"Conclusion"

\[ \frac{1}{2\pi} P_r(t) \, dt \to \delta_0 = \delta - \text{function concentrated at 0}. \]

In our case

\[ \nu \left( r \varphi \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\varphi - t) f(t) \, dt \, \delta_0 \to \delta = t. \]

\[ \to \int f(r \varphi). \text{ so we do expect continuity}. \]

We will prove this rigorously next time.
For functions $g, h : [-\pi, \pi] \to \mathbb{R}$ continuous, set

$$g \ast h(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta - t) h(t) \, dt.$$  

If we write $u_r(\theta) = u(\cos \theta)$ and write $f(t)$ instead of $f(e^{it})$, we obtain

$$u_r = P_r \ast f.$$ Thus we defined the solution to the Dirichlet problem as a convolution.