Math 220C - Lecture 5

April 7, 2021

Last time (Dirichlet Problem)

Given $f : \partial\Delta \longrightarrow \mathbb{R}$ continuous, define

$$u(re^{i\theta}) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})\, dt & , \ r < 1 \\ f(e^{i\theta}) & , \ r = 1 \end{cases}$$

We have seen u harmonic in Δ & $u\big/_{\partial\Delta} = f$.

We show u continuous in $\bar{\Delta}$.

Conclusion u solves the Dirichlet Problem. in $\Delta = \Delta(0,1)$.

Theorem $u : \overline{\Delta} \longrightarrow \mathbb{R}$ is continuous.

Proof The only issue is continuity over $\partial\Delta$ since u is continuous in Δ, being harmonic. We show

$$\lim_{\substack{r \to 1 \\ \theta \to \theta_0}} u(re^{i\theta}) = f(e^{i\theta_0}) \quad \forall\, \theta_0.$$

Claim WLOG $\theta_0 = 0$

Else, rotate! Let

$\tilde{f}(z) = f(e^{i\theta_0} z)$. Let $\tilde{u}$ be the similar function with $\tilde{f}$ instead of f. By the explicit integral & change of variables

$$\tilde{u}(z) = u(e^{i\theta_0} z).$$

Thus u continuous at $\theta_0 \iff \tilde{u}$ is continuous at 1.

Let $\theta_0 = 0$ from now on.

Fix $\varepsilon > 0$. We show $\exists\, \rho, \delta > 0$ such that

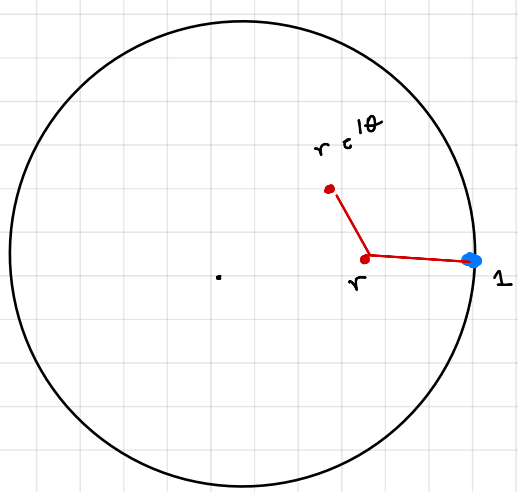
(1) $|u(re^{i\theta}) - u(r)| < \varepsilon$ if $|\theta| < \delta$, all r.

(2) $|u(r) - f(1)| < 2\varepsilon$ if $\rho < r \leq 1$.

Therefore (1) + (2), & triangle inequality gives

$$|u(re^{i\theta}) - f(1)| < 3\varepsilon \quad \forall\, |\theta| < \delta, \quad \rho < r \leq 1.$$

$$\Longrightarrow \lim_{\substack{r \to 1 \\ \theta \to 0}} u(re^{i\theta}) = f(1) \text{ as needed.}$$



Proof of (1)

Since $f : \partial\Delta \longrightarrow \mathbb{R}$ uniformly continuous, let δ such that

$$|x-y| < \delta \implies |f(e^{ix}) - f(e^{iy})| < \varepsilon. \quad (*)$$

We estimate

$$|u(re^{i\theta}) - u(r)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(\theta-t) f(e^{it})\,dt - \int_{-\pi}^{\pi} P_r(-t) f(e^{it})\,dt\right|$$

$$= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(-t) f(e^{it+i\theta})\,dt - \int_{-\pi}^{\pi} P_r(-t) f(e^{it})\,dt\right|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\, \underbrace{|f(e^{it+i\theta}) - f(e^{it})|}_{<\varepsilon \text{ if } |\theta|<\delta \text{ by } (*)}\,dt$$

$$\leq \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(-t)\,dt}_{1 \text{ by Lecture 4}} \cdot \varepsilon = \varepsilon$$

Proof of (2)

$$|u(r) - f(1)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t)\, f(e^{it})\, dt - f(1) \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t)\, f(e^{it})\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t)\, f(1)\, dt \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t)\, \left| f(e^{it}) - f(1) \right| dt$$

If $|t| < \delta \Rightarrow |f(e^{it}) - f(1)| < \varepsilon.$ by (*)

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(-t)\, \left| f(e^{it}) - f(1) \right| dt \leq \varepsilon \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(-t)\, dt$$

$$\leq \varepsilon \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(-t)\, dt = \varepsilon. \quad \text{(Lecture 4)}$$

If $|t| \geq \delta$. Since f continuous $\Rightarrow |f| \leq M$ over $\partial\Delta$.

$$\frac{1}{2\pi} \int_{|t|>\delta} P_r(-t)\, \underbrace{\left| f(e^{it}) - f(1) \right|}_{2M} dt \leq \frac{2M}{2\pi} \int_{|t|>\delta} \underbrace{P_r(-t)}_{\frac{\varepsilon}{2M}} dt$$

$$\leq \frac{2M}{2\pi} \cdot \frac{\varepsilon}{2M} \cdot 2\pi = \varepsilon$$

We used that

$P_r(\pm t) \rightrightarrows 0$ as $r \longrightarrow 1$, in $[\delta, \pi]$ by Lecture 4. Thus $\exists \rho$

$$P_r(\pm t) < \frac{\varepsilon}{2M} \quad \forall\ t \in [\delta, \pi] \text{ and } \rho \leq r \leq 1.$$

Thus $|u(r) - f(1)| < 2\varepsilon. \quad \forall\ \rho \leq r \leq 1.$

Corollary The Dirichlet Problem can be solved in any disc $\Delta(a, R)$.

Why? This follows via translation & dilation

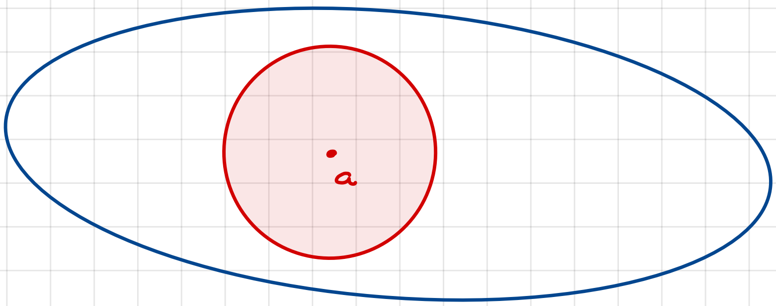
$$z \longrightarrow a + Rz.$$

mapping $\Delta(0,1) \longrightarrow \Delta(a, R)$. We solved the case of $\Delta(0,1)$. above.

Corollary (Converse to MVP)

If $u : G \longrightarrow \mathbb{R}$ continuous & satisfies MVP $\Rightarrow$ u harmonic

Proof



Let $a \in G$. Let $\overline{\Delta}(a, R) \subseteq G$. We show u harmonic in $\Delta(a, R)$.

Let $f = u\big/_{\partial\Delta(a,R)}$. Solve Dirichlet Problem in $\overline{\Delta}(a, R)$.

Thus h harmonic in $\Delta(a, R)$, continuous in $\overline{\Delta}(a, R)$. & $h\big/_{\partial\Delta(a,R)} = f$.

Let $\Phi = h - u : \overline{\Delta}(a, R) \longrightarrow \mathbb{R}$. $\Rightarrow$ $\Phi\big/_{\partial\Delta(a,R)} = 0$ & Φ continuous & satisfies MVP (because h, u do). Then $\Phi \equiv 0$ by Corollary to MP$^+$ (Lecture 2). Thus $u = h =$ harmonic in $\Delta(a, R)$.

II. Convergence of harmonic functions   Conway X.2.

The natural notion of convergence for harmonic functions is local uniform convergence.

Lemma

If $u_n : G \longrightarrow \mathbb{R}$ harmonic & $u_n \overset{l.u.}{\rightrightarrows} u$ then $u : G \to \mathbb{R}$ harmonic.

Proof Since u_n harmonic $\Rightarrow$ u_n continuous $\Rightarrow$ u continuous.

Since u_n harmonic $\Rightarrow$ u_n satisfies MVP. Let $\overline{\Delta}(a,R) \subseteq G$.

$$u_n(a) = \frac{1}{2\pi}\int_0^{2\pi} u_n(a + Re^{it})\, dt$$

Make $n \to \infty$.

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + Re^{it})\, dt$$

$\Rightarrow$ u satisfies MVP. $\Rightarrow$ u harmonic.

We have stronger results

Harnack's Theorem Let $u_n : G \longrightarrow \mathbb{R}$ harmonic. and

$$u_1 \le u_2 \le \dots \le u_n \le \dots \text{ in } G.$$ Then either

(1) $u_n \overset{\text{l.u.}}{\rightrightarrows} u$ & u harmonic. or

(2) $u_n \overset{\text{l.u.}}{\rightrightarrows} \infty$.

Remark If $a_1 \le a_2 \le \dots \le a_n \le \dots$ are real numbers, then

(1) $a_n \longrightarrow a$ where $a = \sup_n a_n < \infty$ or

(2) else $a_n \longrightarrow \infty$