Math 220C - Lecture 6

April 9, 2021

1. Harnack's Theorem Let $u_n : G \longrightarrow \mathbb{R}$ harmonic and

$u_1 \le u_2 \le \dots \le u_n \le \dots$ in G. Then either

(1) $u_n \overset{\ell.u.}{\rightrightarrows} u$ & u harmonic. or

(2) $u_n \overset{\ell.u.}{\rightrightarrows} \infty$.

Remark If $a_1 \le a_2 \le \dots \le a_n \le \dots$ are real numbers,

then either

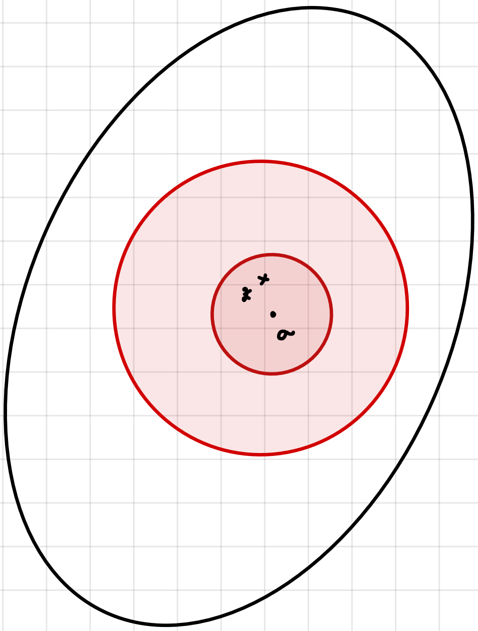
(1) $a_n \longrightarrow a < \infty$

(2) $a_n \longrightarrow \infty$

Remark Harnack Inequality (Lecture 3)

$v : G \longrightarrow \mathbb{R}$, $v \geq 0$, harmonic, $\overline{\Delta}(a, R) \subseteq G$.

If $|z - a| = r < R$, then



$$v(a) \frac{R-r}{R+r} \leq v(z) \leq v(a) \frac{R+r}{R-r}$$

If $r \leq R/2$. $\Rightarrow$ $\frac{R+r}{R-r} \leq 3$, $\frac{1}{3} \leq \frac{R-r}{R+r}$.

If $z \in \Delta\left(a, \frac{R}{2}\right)$ then $\frac{1}{3} v(a) \leq v(z) \leq 3 v(a)$.

Proof of Harnack's theorem

WLOG $u_n \geq 0$. else work with $\tilde{u}_n = u_n - u_1 \geq 0$

Step 1 Pointwise convergence.

Since $\{u_n(z)\}$ is non decreasing $\forall z \in G \Rightarrow$

$\Rightarrow$ either $u_n(z) \longrightarrow \infty$ or $u_n(z) \longrightarrow u(z)$ for some $u(z) < \infty$.

Let $A = \{z \in G : u_n(z) \longrightarrow \infty\}$ $\Rightarrow A \cap B = \Phi$, $A \cup B = G$.

$B = \{z \in G : u_n(z) \longrightarrow u(z)\}$

It suffices to show A, B open. Since G connected $\Rightarrow$

$$A = G \text{ or } B = G.$$

Let $a \in G$. Let $\bar{\Delta}(a, R) \subseteq G$. Let $z \in \bar{\Delta}(a, \frac{R}{2})$.

$$\Rightarrow \frac{1}{3} u_n(a) \leq u_n(z) \leq 3\, u_n(a).$$

$\boxed{i}$ If $a \in A \Rightarrow u_n(a) \longrightarrow \infty$. $\Rightarrow u_n(z) \longrightarrow \infty$. $\Rightarrow z \in A$

$$\Rightarrow \Delta(a, \tfrac{R}{2}) \subseteq A. \Rightarrow A \text{ open}$$

$\boxed{ii}$ If $a \in B \Rightarrow u_n(a) \longrightarrow u(a) < \infty$. $\Rightarrow u_n(z) \longrightarrow u(z) < \infty \Rightarrow z \in B$.

$$\Rightarrow \Delta(a, \tfrac{R}{2}) \subseteq B. \Rightarrow B \text{ open}.$$

Step 2 Local uniform convergence

Let $a \in G$. Let $\overline{\Delta}(a, R) \subseteq G$. We show uniform convergence in $\Delta(a, \frac{R}{2})$. We have two cases:

$\boxed{i}$ $u_n(a) \longrightarrow \infty$. $\Rightarrow \forall M \;\exists N : u_n(a) \geq 3M$ for $n \geq N$

$$\Rightarrow u_n(z) \geq \frac{1}{3} u_n(a) \geq M \quad \forall n \geq N, \; z \in \Delta\left(a, \tfrac{R}{2}\right).$$

$$\Rightarrow u_n \rightrightarrows \infty \quad \text{in} \quad \Delta\left(a, \tfrac{R}{2}\right).$$

$\boxed{ii}$ $u_n(a) \longrightarrow u(a)$. Fix $\varepsilon > 0$. Since $\{u_n(a)\}$ Cauchy

$$\Rightarrow \exists N : 0 \leq u_n(a) - u_m(a) < \frac{\varepsilon}{3} \quad \forall n \geq m \geq N$$

$$\Rightarrow 0 \leq u_n(z) - u_m(z) < 3\left(u_n(a) - u_m(a)\right) < \varepsilon \quad \forall n \geq m \geq N.$$

Make $n \longrightarrow \infty$ $\Rightarrow$ $0 \leq u(z) - u_m(z) \leq \varepsilon \quad \forall m \geq N, \; z \in \Delta\left(a, \frac{R}{2}\right).$

$$\Rightarrow u_m \rightrightarrows u \quad \text{in} \quad \Delta\left(a, \tfrac{R}{2}\right).$$

2. Subharmonic Functions   Conway X. 3.

SH functions share many properties with harmonic fns.

Definition $\varphi : G \longrightarrow \mathbb{R}$ continuous, $\forall\, a \in G$, $\exists\, \overline{\Delta}(a, R) \subseteq G$.

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{it})\, dt \qquad \forall\ 0 \leq r \leq R$$

then φ is called subharmonic.

Superharmonic functions satisfy the opposite inequality.

Remark

[i] φ subharmonic $\Rightarrow -\varphi$ superharmonic

[ii] φ harmonic $\Rightarrow \varphi$ sub/superharmonic

[iii] φ is C^2 & $\Delta \varphi \geq 0 \Rightarrow \varphi$ subharmonic.

This is HWK 2, Problem 1.

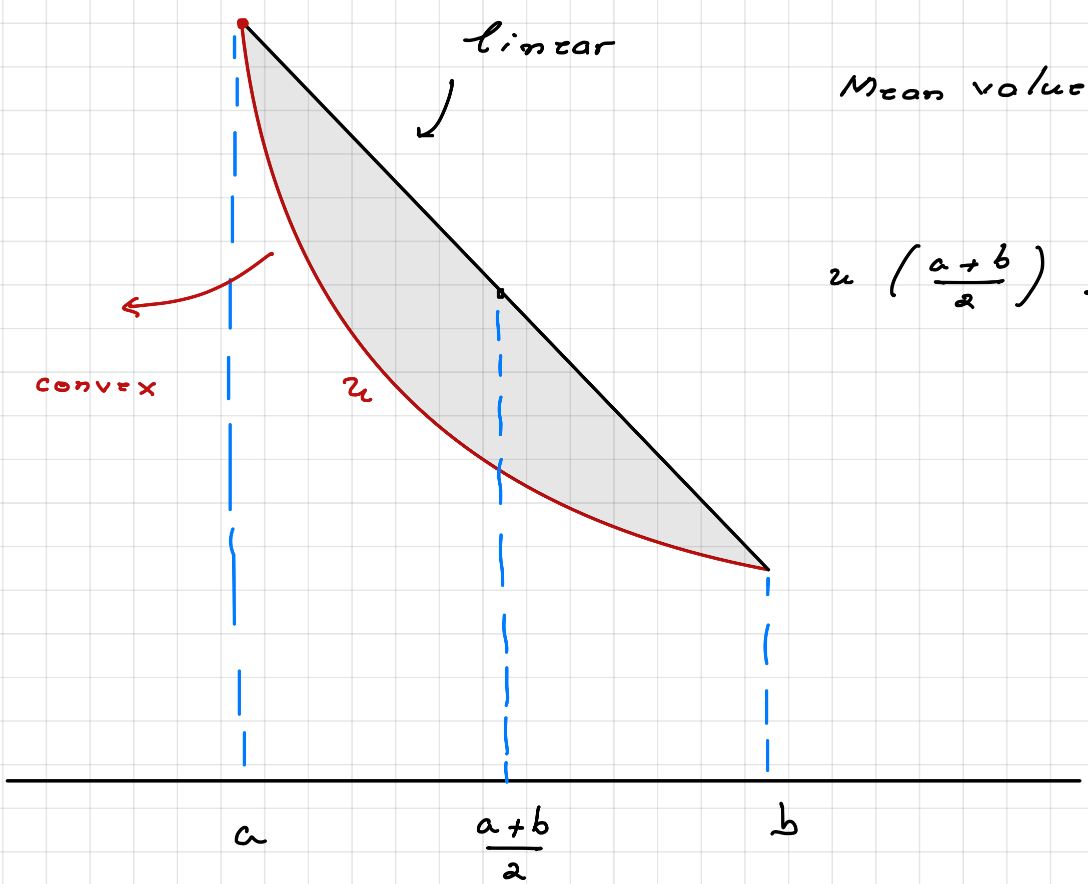
Analogy with 1 real variable

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \longleftrightarrow \frac{\partial^2}{\partial x^2} \quad (1 \text{ variable})$$

"harmonic" $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u$ linear, $u = ax + b$.

C^2 "subharmonic" $\frac{\partial^2 u}{\partial x^2} \geq 0 \Rightarrow u$ convex

C^2 "superharmonic" $\frac{\partial^2 u}{\partial x^2} \leq 0 \Rightarrow u$ concave



Mean value Inequality becomes

$$u\left(\frac{a+b}{2}\right) \leq \frac{u(a) + u(b)}{2}$$

Properties of subharmonic functions

(1) similar to harmonic functions

(2) new properties

---

(1) Maximum principle $\varphi : G \to \mathbb{R}$ subharmonic

MP : If $\varphi : G \to \mathbb{R}$ SH and achieves a maximum at $a \in G$

$\Rightarrow \varphi$ constant.

MP$^+$: If $\varphi : G \to \mathbb{R}$ SH and $\forall\, a \in \partial_\infty G$,

$$\limsup_{z \to a} \varphi(z) \le 0 \Rightarrow \varphi < 0 \text{ or } \varphi \equiv 0 \text{ in } G.$$

Corollary $\varphi : \bar{G} \to \mathbb{R}$, G bounded, φ continuous in $\bar{G}$, subharmonic in G, $\varphi|_{\partial G} \le 0$. Then $\varphi < 0$ or $\varphi \equiv 0$ in G.

(2) New Property

φ_1, φ_2 subharmonic $\Rightarrow \varphi = \max(\varphi_1, \varphi_2)$ subharmonic.

Proof φ continuous. If $a \in G$, we can find R_1, R_2 with

$$\varphi_1(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi_1(a + re^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt \quad \forall\, r < R_1$$

$$\varphi_2(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi_2(a + re^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{it})\,dt \quad \forall\, r < R_2$$

For $R = \min(R_1, R_2)$, $r < R$ we have

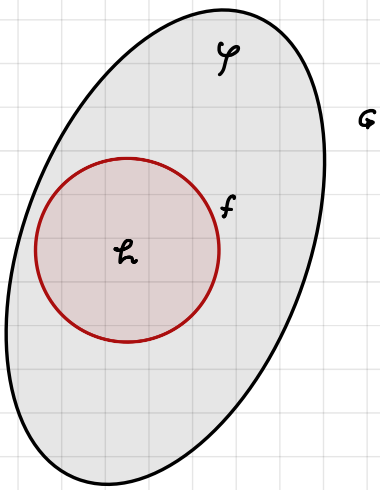
$$\varphi(a) = \max(\varphi_1(a), \varphi_2(a)) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{it})\,dt$$

$\Rightarrow \varphi$ subharmonic.

New Property (Poisson Modification / Bumping)

φ subharmonic $\Rightarrow$ $\tilde{\varphi}$ subharmonic

Construction



Let φ be subharmonic.

Let $\overline{\Delta}(a,r) \subseteq G$. Let $f = \varphi|_{\partial\Delta}$.

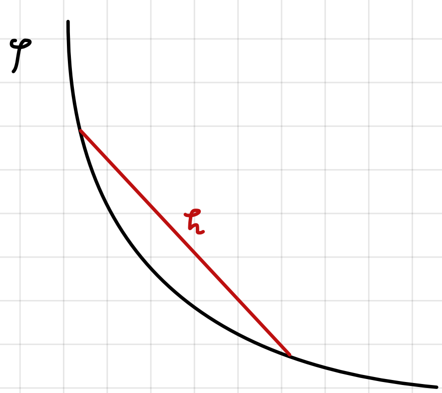
Let h be the solution to Dirichlet in Δ,

boundary value f $\Rightarrow$ $h|_{\partial\Delta} = f$.

Define

$$\tilde{\varphi} = \begin{cases} \varphi & \text{in } G \setminus \overline{\Delta} \\ h & \text{in } \Delta. \end{cases}$$

In one variable



Claims

[i] $\varphi \leq \tilde{\varphi}$

[ii] $\tilde{\varphi}$ subharmonic

[iii] $\varphi \leq \psi \implies \tilde{\varphi} \leq \tilde{\psi}$

We will prove these statements next time.