Homework 3 posted. There are 6 questions.

Questions 2 - 5 are about the Dirichlet Problem.

Last time (Poisson modification / Bumping)

- \( \varphi : \mathbb{C} \to \mathbb{R} \) subharmonic
- \( \overline{\Delta} \subseteq \mathbb{C} \) closed disc
- \( f = \frac{\varphi}{\partial \Delta} \)
- Solve Dirichlet Problem in \( \overline{\Delta} \):
  - \( h \) continuous in \( \overline{\Delta} \), harmonic in \( \Delta \), \( \frac{\partial h}{\partial \Delta} = f \)
- Let \( \tilde{\varphi} = \begin{cases} \varphi & \text{in } \mathbb{C} \setminus \Delta \\ h & \text{in } \overline{\Delta} \end{cases} \Rightarrow \tilde{\varphi} \text{ cont.} \)
Proposition Conway 3.7

1. $\tilde{y} \leq \tilde{y}$

2. $\tilde{y}$ subharmonic (HWK 3)

3. $\tilde{y}_1 \leq \tilde{y}_2$ subharmonic $\Rightarrow$ $\tilde{y}_1 \leq \tilde{y}_2$

Proof

Since $y = \tilde{y}$ in $G \setminus \bar{\Omega}$, we only need to prove $y \leq \tilde{h}$ in $\bar{\Omega}$.

Note that $y - h$ is subharmonic ($y$ satisfies MV inequality, $h$ satisfies MV equality). Note

$\frac{y - h}{\partial \Omega} = f - f = 0$.

By Maximum Principle $y - h \leq 0$ in $\bar{\Omega}$, as needed.
Let \( f_1 = \frac{\partial \psi_1}{\partial n} \) and \( f_2 = \frac{\partial \psi_2}{\partial n} \).

Let \( h_1, h_2 : \bar{\Omega} \rightarrow \mathbb{R} \) solve Dirichlet Problem with boundary values \( f_1, f_2 \).

To show \( \psi_1 \leq \psi_2 \), it suffices to show \( h_1 \leq h_2 \) in \( \bar{\Omega} \).

Note \( h_1 - h_2 \) harmonic in \( \Omega \), continuous in \( \bar{\Omega} \)

\[
\frac{h_1 - h_2}{\partial n} = f_1 - f_2 = \frac{\partial \psi_1}{\partial n} - \frac{\partial \psi_2}{\partial n} \leq 0
\]

By Maximum Principle, \( h_1 - h_2 \leq 0 \) in \( \bar{\Omega} \) as needed.
Question: How do we construct interesting harmonic functions?

Methods:
1. \( u = \Re f, \ f \) holomorphic
2. Poisson's formula / Dirichlet Problem, \( G = \Delta \)
3. Perron method

Idea behind Perron's method - 1 variable

Let \( \gamma \) be a convex path joining \( A, B \).

Note: \( \gamma \leq u \) where \( u = \) line.

Note:
\[
    u(2) = \sup \{ \gamma(z) : \gamma \text{ convex as above} \}.
\]

We wish to extend these observations to \( \mathbb{R}^2 \).
In \( \mathbb{R}^2 \): \( G \subseteq \mathbb{R} \) bounded, \( f: G \rightarrow \mathbb{R} \) continuous

- Perron family

\[ P(G, f) = \{ \varphi: G \rightarrow \mathbb{R} \text{ subharmonic}, \limsup_{z \to a} \varphi(z) \leq f(a), \forall a \in G \} \]

- Perron function \( u: G \rightarrow \mathbb{R} \)

\[ u(z) = \sup \{ \varphi(z), \varphi \in P(G, f) \} \]

**Question** Is the Perron function well-defined?
Remarks

\[ P(c, f) \neq \emptyset. \]

Indeed, \( \exists c \) compact, \( f \) cont. \( \Rightarrow m \leq f \leq M \) in \( \mathcal{E} \).

Then \( \varphi = m \) is in \( P(c, f) \).

\[ u \text{ is well-defined.} \]

Since \( f \leq M \), we have

\[
\limsup_{x \to a} \varphi(x) \leq M \quad \forall a \in \mathcal{E} \Rightarrow \varphi \leq M \text{ by MP}
\]

\( \Rightarrow u(a) = \sup \{ \varphi(x) \} \leq M. \)

\[ \varphi \in P(c, f) \Rightarrow \tilde{\varphi} \in P(c, f) \]

Indeed, \( \tilde{\varphi} \) subharmonic (see Proposition) and \( \varphi = \tilde{\varphi} \) near \( a \in \mathcal{E} \) so

\[
\limsup_{x \to a} \tilde{\varphi}(x) = \limsup_{x \to a} \varphi(x) \leq f(a). \]
Theorem Conway 3.11.

The Perron function $u$ is harmonic.

Proof. Let $x \in G$, $\Delta \subseteq G$ a disc around $x$.

WTS $u$ harmonic in $\Delta$.

Step 1. Find functions $f_n \in \mathcal{P}(G,f)$ with $f_n(x) \to u(x)$.

This is possible by the definition of $u$.

WLOG we may assume $f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots$

Why? Else, define

$$y_n^{\text{new}} = \max (f_1, f_2, \ldots, f_n).$$

By Lecture 6, Property (2), $y_n^{\text{new}} \in \mathcal{P}(G,f)$. Note that

$$y_n^{\text{new}}(x) \to u(x)$$
as well and that

$$f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots$$

We drop the superscript "new" from now on.
Step 2 WLOG We may assume

\[ y_1 \leq y_2 \leq \ldots \leq y_n \leq \ldots \text{ are harmonic in } \Delta. \]

Why? Indeed,

\[ \tilde{y}_1 \leq \tilde{y}_2 \leq \ldots \leq \tilde{y}_n \leq \ldots \] (see Proposition above)

and \( \tilde{y}_n \in \mathcal{P}(c, f) \) by Remark 1. Note \( \tilde{y}_n \) are harmonic in \( \Delta \).

Furthermore \( \tilde{y}_n(x) \to u(x) \) still holds. Indeed, \( y_n(x) \leq \tilde{y}_n(x) \leq u(x) \)

Thus \( y_n(x) \to u(x) \implies \tilde{y}_n(x) \to u(x) \)

We can work with the functions \( \tilde{y}_n \) instead of the \( y_n \)'s.
Step 3  By Harnack’s convergence

\[ \tilde{\varphi}_n \xrightarrow{\text{u.u.}} U \text{ in } \Delta, \text{ for } U \text{ harmonic.} \]

We noted that \( \tilde{\varphi}_n (x) \to u(x) < \infty \) so the possibility \( \tilde{\varphi}_n \xrightarrow{\text{u.u.}} \infty \) in Harnack is not allowed.

Note: \( \tilde{U}(x) = u(x) \).

Goal: We show \( \tilde{U} = u \text{ in } \Delta \). (not only at \( x \)).

This will show \( u \) is harmonic, as needed.
Step 4 Let \( y \in \Omega \). We show \( U(y) = u(y) \).

Let \( y_n \in \mathcal{P} \), \( \psi_n(y) \rightarrow u(y) \), possible by definition of \( u \).

\textit{WLOG} \quad \varphi_n \preceq \varphi_n

\textit{Why?} \quad \varphi_n^{\text{new}} = \max (\varphi_n, \varphi_n) = \text{subharmonic (Lecture 6)}.

We still have \( \varphi_n^{\text{new}} \in \mathcal{P}(c, f) \).

\textit{Bonus} \quad \varphi_n(x) \rightarrow u(x) \land \varphi_n(y) \rightarrow u(y).

\textit{Why?} \quad \text{We know } \varphi_n(x) \rightarrow u(x) \quad \text{and} \quad \varphi_n(y) \rightarrow u(y).

\[
\varphi_n \preceq \varphi_n^{\text{new}} \preceq u \quad \text{def. of } \varphi_n^{\text{new}}
\]

The same argument works for \( y \), with \( \varphi_n \) instead of \( \varphi_n \).
We run the above steps for $\psi_1, \psi_2, \ldots$

**Step 1**

$\psi_1 \leq \psi_2 \leq \ldots \leq \psi_n \leq \ldots$

**Step 2**

$\psi_1 \leq \tilde{\psi}_2 \leq \ldots \leq \tilde{\psi}_n \leq \ldots$ harmonic in $\Delta$

**Step 3**

Harnack $\tilde{\psi}_n \mapsto V$ in $\Delta$.

**Claims**

1. $V(y) = \lim_{n \to \infty} \tilde{\psi}_n(y) = u(y)$
2. $U(x) = \lim_{n \to \infty} \tilde{\psi}_n(x) = u(x)$
3. $U(x) = \lim_{n \to \infty} \tilde{\psi}_n(x) = \lim_{n \to \infty} \tilde{\psi}_n(x) = \tilde{V}(x)$

using that $\psi_n \leq \psi_n$ and $\tilde{\psi}_n \leq \tilde{\psi}_n$. 
Conclusion

We know $U - V \leq 0$ in $\Delta$ by (4).

$$U(x) = u(x) = V(x) \Rightarrow (U - V)(x) = 0.$$  
Steps (5)

By Max. Principle $\Rightarrow U - V \equiv 0.$

In particular, $U(y) = V(y) = u(y).$

Since $y \in \Delta$ is arbitrary $\Rightarrow U \equiv u$, as needed.
Oskar Perron (1880 – 1975) was a German mathematician.

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Contributions to partial differential equations, the Perron method.