Math 220C - Lecture 7

April 12, 2021

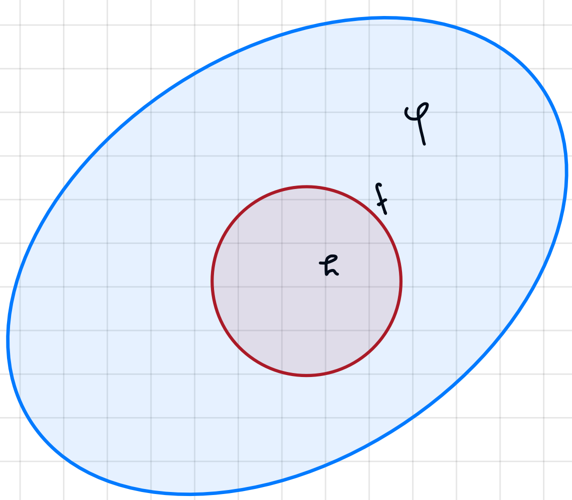
Homework 3 posted. There are 6 questions.

Questions 2 − 5 are about the Dirichlet Problem.

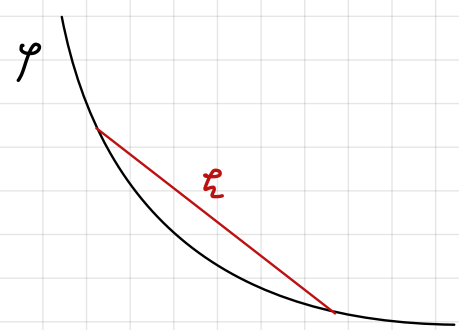
Last time (Poisson modification / Bumping)

- $\varphi : G \longrightarrow \mathbb{R}$ subharmonic
- $\overline{\Delta} \subseteq G$ closed disc
- $f = \varphi|_{\partial \Delta}$.
- Solve Dirichlet Problem in $\overline{\Delta}$:

  h continuous in $\overline{\Delta}$, harmonic in Δ, $h|_{\partial\Delta} = f$.
- Let $\tilde{\varphi} = \begin{cases} \varphi & \text{in } G \setminus \overline{\Delta} \\ h & \text{in } \overline{\Delta} \end{cases} \Rightarrow \tilde{\varphi}$ cont.



in $\mathbb{R}^2$



in $\mathbb{R}$.

Proposition Conway 3.7$^+$

$\boxed{i}$ $\varphi \le \tilde{\varphi}$

$\boxed{ii}$ $\tilde{\varphi}$ subharmonic (HWK 3)

$\boxed{iii}$ $\varphi_1 \le \varphi_2$ subharmonic $\Longrightarrow$ $\tilde{\varphi}_1 \le \tilde{\varphi}_2$.

Proof $\boxed{i}$ Since $\varphi = \tilde{\varphi}$ in $G \setminus \bar{\Delta}$, we only need to prove

$$\varphi \le h \text{ in } \bar{\Delta}.$$

Note that $\varphi - h$ is subharmonic (φ satisfies MV-inequality, h satisfies MV-equality). Note

$$\varphi - h \big|_{\partial\Delta} = f - f = 0.$$

By Maximum Principle $\varphi - h \le 0$ in $\bar{\Delta}$, as needed.

$\boxed{iii}$ Let $f_1 = \varphi_1\big|_{\partial\Delta}$, $f_2 = \varphi_2\big|_{\partial\Delta}$.

Let $h_1, h_2 : \bar{\Delta} \longrightarrow \mathbb{R}$ solve Dirichlet Problem with boundary values f_1, f_2.

To show $\tilde{\varphi}_1 \le \tilde{\varphi}_2$, it suffices to show $h_1 \le h_2$ in $\bar{\Delta}$.

Note $h_1 - h_2$ harmonic in Δ, continuous in $\bar{\Delta}$

$$h_1 - h_2\big|_{\partial\Delta} = f_1 - f_2 = \varphi_1\big|_{\partial\Delta} - \varphi_2\big|_{\partial\Delta} \le 0$$

By Maximum Principle, $h_1 - h_2 \le 0$ in $\bar{\Delta}$ as needed.

Question How do we construct interesting harmonic functions?

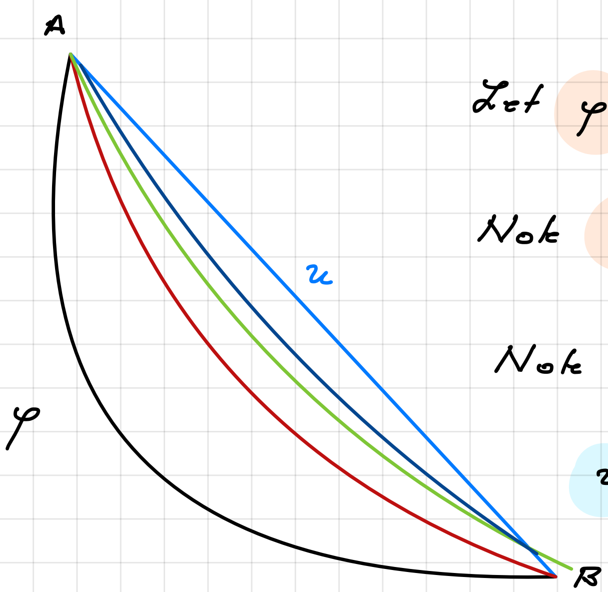
Methods

$\boxed{i}$ $u = \mathrm{Re}\, f$, f holomorphic

$\boxed{ii}$ Poisson's formula / Dirichlet Problem, $G = \Delta$

$\boxed{iii}$ Perron method

Idea behind Perron's method – 1 variable



Let φ be a convex path joining A, B.

Note $\varphi \leq u$ where $u =$ line.

Note

$$u(z) = \sup\{\varphi(z) : \varphi \text{ convex as above}\}.$$

We wish to extend these observations to $\mathbb{R}^2$.

In $\mathbb{R}^2$: $G \subseteq \mathbb{C}$ bounded, $f : \partial G \longrightarrow \mathbb{R}$ continuous

- **Perron family**

$$\mathcal{P}(G, f) = \left\{ \varphi : G \longrightarrow \mathbb{R} \text{ subharmonic}, \ \limsup_{z \to a} \varphi(z) \leq f(a) \ \forall\, a \in \partial G. \right\}.$$

- **Perron function** $u : G \longrightarrow \mathbb{R}$

$$u(z) = \sup \left\{ \varphi(z),\ \varphi \in \mathcal{P}(G, f) \right\}$$

Question Is the Perron function well-defined?

Remarks $\boxed{I}$ $\mathcal{P}(G, f) \neq \emptyset$.

Indeed, ∂G compact, f cont. $\Longrightarrow$ $m \leq f \leq M$ in ∂G.

Then $\varphi = m$ is in $\mathcal{P}(G, f)$.

$\boxed{II}$ u is well-defined.

Since $f \leq M$, we have

$$\limsup_{z \to a} \varphi(z) \leq M \quad \forall\, a \in \partial G \implies \varphi \leq M \text{ by MP}$$

$$\implies u(z) = \sup \{ \varphi(z) \} \leq M.$$

$\boxed{III}$ $\varphi \in \mathcal{P}(G, f) \implies \tilde{\varphi} \in \mathcal{P}(G, f)$

Indeed, $\tilde{\varphi}$ subharmonic (see Proposition) and $\varphi = \tilde{\varphi}$ near $a \in \partial G$ so

$$\limsup_{z \to a} \tilde{\varphi}(z) = \limsup_{z \to a} \varphi(z) \leq f(a).$$

Theorem Conway 3.11.

The Perron function u is harmonic

Proof Let $x \in G$, $\overline{\Delta} \subseteq G$ a disc around x.

WTS u harmonic in Δ.

Step 1 Find functions $\varphi_n \in \mathcal{P}(G, f)$ with $\varphi_n(x) \longrightarrow u(x)$.

This is possible by the definition of u.

WLOG we may assume $\varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_n \leq \dots$

Why? Else, define

$$\varphi_n^{new} = \max(\varphi_1, \varphi_2 \dots \varphi_n).$$

By Lecture 6, Property (2), $\varphi_n^{new} \in \mathcal{P}(G, f)$. Note that

$\varphi_n^{new}(x) \longrightarrow u(x)$ as well and that

$$\varphi_1^{new} \leq \varphi_2^{new} \leq \dots \leq \varphi_n^{new} \leq \dots$$

We drop the superscript "new" from now on.

Step 2 **WLOG** We may assume

$$\varphi_1 \le \varphi_2 \le \dots \le \varphi_n \le \dots \quad \text{are harmonic in } \Delta.$$

Why? Indeed,

$$\tilde{\varphi}_1 \le \tilde{\varphi}_2 \le \dots \le \tilde{\varphi}_n \le \dots \quad (\text{see Proposition above})$$

and $\tilde{\varphi}_n \in \mathcal{P}(G, f)$ by Remark [iii]. Note $\tilde{\varphi}_n$ are harmonic in Δ.

Furthermore $\tilde{\varphi}_n(x) \longrightarrow u(x)$ still holds. Indeed,

Proposition [I] definition of u as supremum

$$\varphi_n(x) \le \tilde{\varphi}_n(x) \le u(x)$$

Thus $\varphi_n(x) \longrightarrow u(x) \implies \tilde{\varphi}_n(x) \longrightarrow u(x)$

We can work with the functions $\tilde{\varphi}_n$ instead of the φ_n's.

Step 3 By Harnack's convergence

$$\tilde{\varphi}_n \overset{l.u.}{\rightrightarrows} U \text{ in } \Delta, \text{ for } U \text{ harmonic.}$$

We noted that $\tilde{\varphi}_n(x) \longrightarrow u(x) < \infty$ so the possibility

$\tilde{\varphi}_n \overset{l.u.}{\rightrightarrows} \infty$ in Harnack is not allowed.

Note $U(x) = u(x)$.

Goal We show $U = u$ in Δ. (not only at x).

This will show u is harmonic, as needed.

Step 4 Let $y \in \Delta$. We show $U(y) = u(y)$.

Let $\psi_n \in \mathcal{P}$, $\psi_n(y) \longrightarrow u(y)$, possible by definition of u.

WLOG $\varphi_n \leq \psi_n$

Why ? $\psi_n^{new} = \max(\psi_n, \varphi_n) =$ subharmonic (Lecture 6).

We still have $\psi_n^{new} \in \mathcal{P}(G, f)$.

Bonus $\psi_n^{new}(x) \longrightarrow u(x)$ & $\psi_n^{new}(y) \longrightarrow u(y)$.

Why ? We know $\varphi_n(x) \longrightarrow u(x)$

$\varphi_n \leq \psi_n^{new} \leq u$ $\Longrightarrow$ $\psi_n^{new}(x) \longrightarrow u(x)$.

def. of ψ_n^{new}

definition of u as supremum

The same argument works for y, with ψ_n instead of φ_n.

We run the above Steps for $\psi_1, \psi_2, \dots$

Step 1' $\quad \psi_1 \le \psi_2 \le \dots \le \psi_n \le \dots$

Step 2' $\quad \tilde{\psi}_1 \le \tilde{\psi}_2 \le \dots \le \tilde{\psi}_n \le \dots$ harmonic in Δ

Step 3' $\quad$ Harnack $\quad \tilde{\psi}_n \overset{l.u.}{\rightrightarrows} V$ in Δ.

Claims

a $\quad V(y) \overset{\text{Step 3'}}{=} \lim_{n\to\infty} \tilde{\psi}_n(y) \overset{\text{Bonus}}{=} u(y)$

b $\quad V(x) \overset{\text{Step 3'}}{=} \lim_{n\to\infty} \tilde{\psi}_n(x) \overset{\text{Bonus}}{=} u(x)$

c $\quad U(z) \overset{\text{Step 3}}{=} \lim \tilde{\varphi}_n(z) \le \lim \tilde{\psi}_n(z) \overset{\text{Step 3'}}{=} V(z)$

using that $\varphi_n \le \psi_n$ and $\tilde{\varphi}_n \le \tilde{\psi}_n$.

Conclusion

We know $U - V \leq 0$ in Δ by $\boxed{c}$

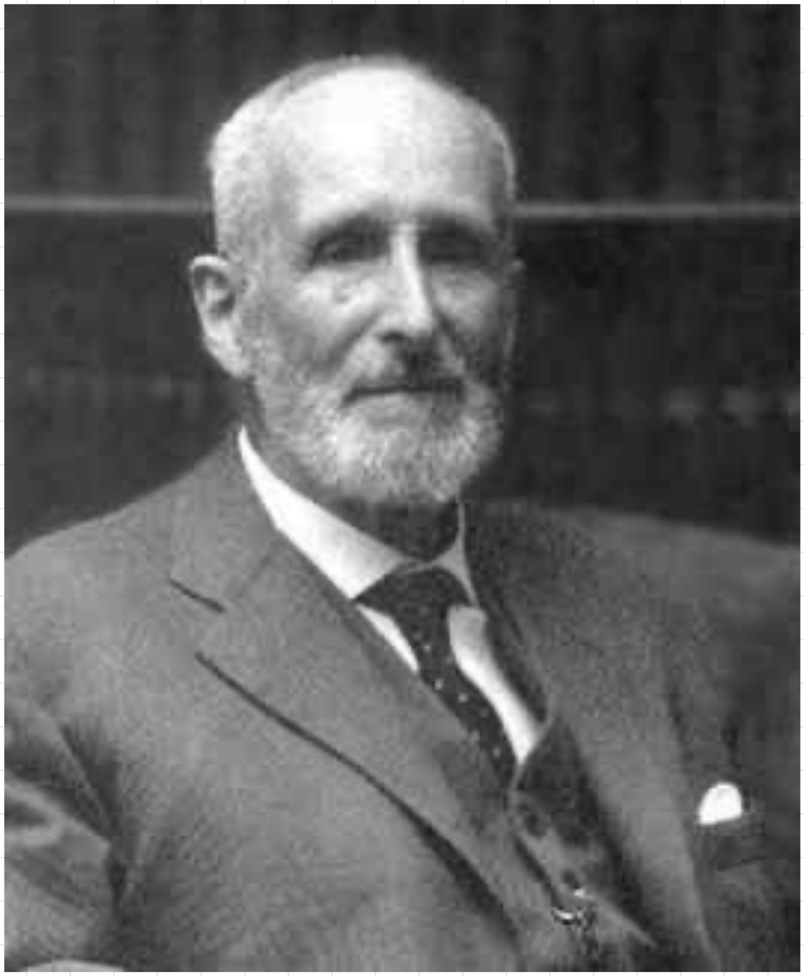
$$U(x) \underset{\text{Step 3}}{=} u(x) \underset{\boxed{b}}{=} V(x) \implies (U-V)(x) = 0.$$

harmonic

By Max. Principle $\implies U - V \equiv 0.$

In particular, $U(y) = V(y) \underset{\boxed{a}}{=} u(y).$

Since $y \in \Delta$ is arbitrary $\implies U \equiv u$, as needed.



Oskar Perron (1880 – 1975) was a German mathematician. He brought contributions to PDE's, known for the Perron method.