Math 220C - Lecture 8

April 14, 2021

Plan — short discussion of Dirichlet Problem

— begin Chp XI — Jensen's formula

Last time G bounded, $f: \partial G \longrightarrow \mathbb{R}$ continuous

- **Perron family**

$$\mathcal{P}(G, f) = \left\{ \varphi : G \longrightarrow \mathbb{R} \text{ subharmonic}, \ \limsup_{z \to a} \varphi(z) \leq f(a) \ \forall \ a \in \partial G \right\}.$$

- **Perron function** $u : G \longrightarrow \mathbb{R}$

$$u(z) = \sup \left\{ \varphi(z), \ \varphi \in \mathcal{P}(G, f) \right\}$$

- **Theorem**

The Perron function u is harmonic

Question Does the Perron function solve Dirichlet Problem?

What is the issue?

We know u is harmonic in G.

We need to show $\lim_{z \to a} u(z) = f(a)$ $\forall\, a \in \partial G$.

Answer (HWK 3, #2) **NO!**

If $G = \Delta(0,1) \setminus \{0\}$, we show that the Dirichlet Problem does not always admit a solution.

Better answer In special cases, it does!

Terminology (differs from Conway $\overline{X}.4$)

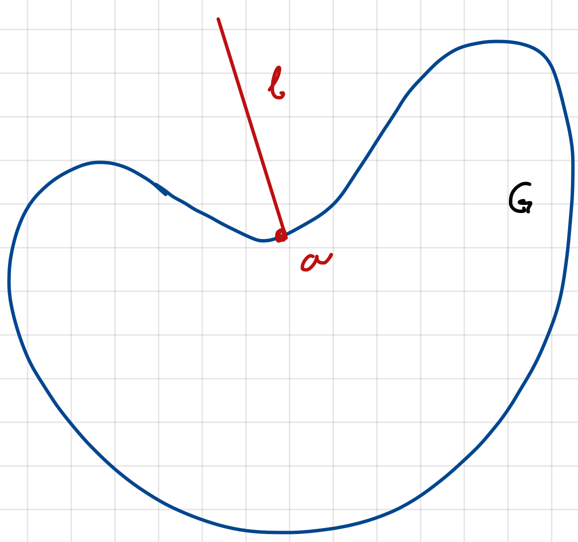
Let G be bounded. Let $a \in \partial G$.

> $\omega : \overline{G} \longrightarrow \mathbb{R}$ continuous in $\overline{G}$, harmonic in G,
>
> $\omega(a) = 0$, $\omega > 0$ in $\partial G \setminus \{a\}$
>
> ω is said to be a **barrier** at a.

The terminology is due to Lebesgue.

Example (HWK 3, # 5) Many reasonable domains satisfy this definition. For instance, if $\exists$ ℓ segment

$\ell \cap \overline{G} = \{a\}$ then there is a barrier at a.



Theorem The Dirichlet Problem can be always be solved in G.

$\iff \forall\, a \in \partial G$, $\exists$ barrier at a.

The Perron function solves the Dirichlet Problem.

Remark „$\Rightarrow$" HWK 3, #4

„$\Leftarrow$" A proof is given in the Appendix to the lecture.

& video on Canvas.

§2. Jensen's Formula

$f : G \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero in G, $\overline{\Delta}(0,r) \subseteq G$.

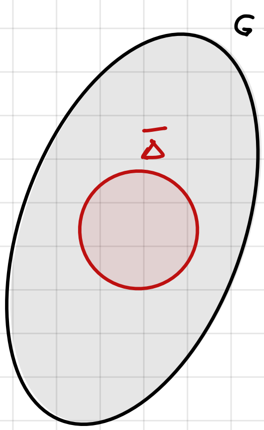
Recall from HWK 1

5. Let $U \subset \mathbb{C}$ be open connected.

(i) Show that if $h : U \to \mathbb{C}$ is holomorphic and nowhere zero in U, then

$$u(z) = \log |h(z)|$$

is harmonic in U.



Mean Value Property for $\log |f|$ gives

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Question What if f has zeros?

The zeros of f will give corrections to the formula.

Theorem $f : G \to \mathbb{C}$ holomorphic, $\overline{\Delta}(0,r) \subseteq G$, $f(0) \neq 0$.

Let $a_1, \dots, a_k$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log|f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})|\, dt.$$

Proof Shrinking G, we may assume $G = \Delta(0,R)$

We may assume $r = 1$. Indeed, otherwise let

$$f^{new}(z) = f(rz) \quad \text{defined in} \quad G^{new} = \Delta\left(0, \frac{R}{r}\right) \supseteq \overline{\Delta}(0,1).$$

When f is holomorphic in $\Delta(0,R) \supseteq \overline{\Delta}(0,1)$, we show

$$\log|f(0)| - \sum_{k=1}^{n} \log|a_k| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{it})|\, dt. \qquad (*)$$

Proof of (*) Let

- $a_1, \dots, a_k$ be zeroes of f in $\Delta = \Delta(0,1)$
- $b_1, \dots, b_m$ be zeroes of f on $\partial\Delta$.

Recall $\varphi_a : \bar{\Delta} \longrightarrow \bar{\Delta}$, $\partial\Delta \longrightarrow \partial\Delta$, $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

Let
$$F(z) = f(z) \Big/ \prod_{j=1}^{k} \varphi_{a_j}(z) \cdot \prod_{j=1}^{m} \frac{b_j}{b_j - z}$$

Note that F has no zeroes in $\bar{\Delta}$ & in fact in a neighborhood of $\bar{\Delta}$. Note

$$F(0) = f(0) \Big/ \prod_{j=1}^{m} (-a_j)$$

By the previous observation applied to F

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{it})| \, dt. \quad (1).$$

By substitution, we find

$$\log|F(0)| = \log|f(0)| - \sum_{j=1}^{k} \log|a_k| \qquad (2)$$

$$\int_0^{2\pi} \log|F(e^{it})|\,dt = \int_0^{2\pi} \log|f(e^{it})|\,dt$$

0 (see below)

$$- \sum_{j=1}^{k} \int_0^{2\pi} \log|\varphi_{a_j}(e^{it})|\,dt$$

0 (claim)

$$+ \sum_{j=1}^{m} \int_0^{2\pi} \log\left|\frac{b_j}{b_j - e^{it}}\right| dt$$

$$= \int_0^{2\pi} \log|f(e^{it})|\,dt. \qquad (3)$$

Here we used $\varphi_{a_j} : \partial\Delta \longrightarrow \partial\Delta$ so that

$$|\varphi_{a_j}(e^{it})| = 1 \implies \log|\varphi_{a_j}(e^{it})| = 0.$$

Jensen's formula follows from (1), (2), (3).

<u>Claim</u> $\displaystyle\int_0^{2\pi} \log\left|\frac{b}{b - e^{it}}\right| dt = 0 \quad \forall\ |b| = 1.$

Proof of the claim Let $b = e^{i\alpha}$. Then

$$\int_0^{2\pi} \log \left| \frac{b}{b - e^{it}} \right| dt = \int_0^{2\pi} \log \left| \frac{e^{i\alpha}}{e^{i\alpha} - e^{it}} \right| dt$$

$$= \int_0^{2\pi} \log \left| \frac{1}{1 - e^{i(t-\alpha)}} \right| dt \qquad t \to t+\alpha$$

$$= \int_0^{2\pi} \log \frac{1}{|1 - e^{it}|} dt$$

$$= - \int_0^{2\pi} \log |1 - e^{it}| \, dt \overset{?}{=} 0 .$$

We note that

$$|1 - e^{it}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t = 4 \sin^2 \frac{t}{2} .$$

We need to show

$$\int_0^{2\pi} \log \left| 2 \sin \frac{t}{2} \right| dt = 0 \quad \overset{t = 2u}{\Longleftrightarrow}$$

$$\Longleftrightarrow \quad \int_0^{\pi} \log |2 \sin u| \, du = 0$$

$$\Longleftrightarrow \quad \int_0^{\pi} \log 2 \, du + \int_0^{\pi} \log \sin u \, du = 0$$

$$\Longleftrightarrow \quad \boxed{\int_0^{\pi} \log \sin u \, du = - \pi \log 2.}$$

Calculation $\displaystyle\int_0^{\pi} \log \sin u \, du = -\pi \log 2.$

Convergence

$$\int_0^{\pi} \log \sin u \, du \le \int_0^{\pi} \log u \, du = u \log u - u \Big|_{u=0}^{u=\pi} < \infty.$$

This uses $\lim\limits_{u \to 0} u \log u = 0.$

Evaluation

$$I = \int_0^{\pi} \log \sin u \, du \overset{u=2v}{=}$$

$$= 2 \int_0^{\pi/2} \log \sin 2v \, dv = \qquad \sin 2v = 2 \sin v \cos v.$$

$$= 2 \int_0^{\pi/2} \log 2 \, dv + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \cos v \, dv$$

$$= \pi \log 2 + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} + v\right) dv$$

$$= \pi \log 2 + 2 \int_0^{\pi} \log \sin v \, dv$$

$$= \pi \log 2 + 2I \implies I = -\pi \log 2.$$

SUR UN NOUVEL ET IMPORTANT THÉORÈME DE LA THÉORIE DES FONCTIONS

PAR

J. L. W. V. JENSEN.

Monsieur le Professeur,

Lors de votre dernier séjour à Copenhague j'ai eu honneur de vous entretenir au sujet d'une intégrale définie appelée, si je ne me trompe, à jouer un rôle dans la théorie des fonctions analytiques. Comme il me parut que cette question vous interéssa vivement, je profiterai de cette occasion — l'envoi des deux petits mémoires[1] destinés à votre Journal — pour vous communiquer le développement détaillé de mon théorème.

Soit $z = re^{\theta i}$ une variable complexe, et α un nombre complexe différent de zéro, on a pour $r < |\alpha|$,

$$l\left(1 - \frac{z}{a}\right) = -\sum_{\nu=1}^{\infty} \frac{1}{\nu}\left(\frac{z}{a}\right)^{\nu}$$

où l désigne la valeur principale du logarithme. En prenant les parties réelles des deux membres et en observant que l'on a $\Re(a) = \frac{1}{2}(a + \dot{a})$,[2] on trouve

$$l\left|1 - \frac{z}{a}\right| = -\sum_{\nu=1}^{\infty} \frac{r^{\nu}}{2\nu}\left(\frac{e^{\nu\theta i}}{a^{\nu}} + \frac{e^{-\nu\theta i}}{\dot{a}^{\nu}}\right), \qquad r = |z| < |\alpha|. \tag{1}$$

[1] (1) *Sur les fonctions entières.*

(2) *Note sur une condition nécessaire et suffisante pour que tous les zéros d'une fonction entière soient réels.*

[2] Ici et dans la suite je désigne toujours par $\Re(a)$ la partie réelle et par $\dot{a}$ la valeur conjuguée de a.

Acta Math 1899, volume 22

Johan Jensen (1859–1925) was a Danish mathematician. He pursued mathematics while worked as a telephone engineer.

Jensen found his formula while unsuccessfully trying to prove the Riemann hypothesis.

He is also known for Jensen's inequality (about convex functions).