Math 220C - Lecture 9

April 16, 2021

Today – Poisson – Jensen formula

– Application of Jensen

– Order of entire functions

Last time

- $f : G \longrightarrow \mathbb{R}$ holomorphic, $f(0) \neq 0$, $\bar{\Delta}(0,r) \subseteq G$
- $a_1, \dots, a_k$ all zeroes of f in $\Delta(0,r)$. w/ multiplicities

$$\log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Question How about values not at the center?

§1. Poisson – Jensen formula

We generalize both

– Jensen's formula & Poisson's formula

Theorem Let $f : G \longrightarrow \mathbb{C}$ holomorphic, $\bar{\Delta}(0,r) \subseteq G$, $z_0 \in \bar{\Delta}(0,r)$. $f(z_0) \neq 0$. Let $a_1, \dots, a_n$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log|f(z_0)| + \underbrace{\sum_{k=1}^{n} \log\left|\frac{r^2 - \bar{a}_k z_0}{r(z_0 - a_k)}\right|}_{\text{contribution from zeroes}} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\operatorname{Re}\frac{re^{it}+z_0}{re^{it}-z_0}}_{\text{Poisson Kernel (Lectures 3 \& 4)}} \cdot \log|f(re^{it})|\,dt$$

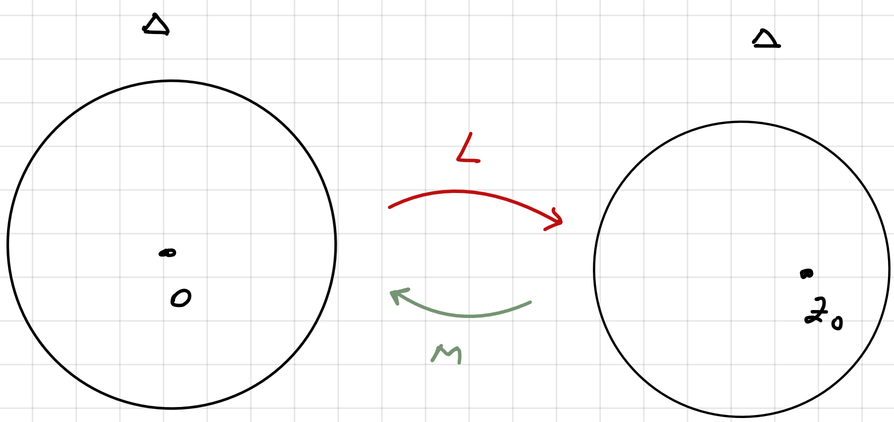
Remark [I] When $z_0 = 0$, we recover Jensen's formula.

[II] If f has no zeroes, this becomes Poisson's formula for the function $\log|f|$, which is harmonic in this case.

Proof WLOG $r=1$. Let $\Delta = \Delta(0,1)$. **WTS**

$$\log|f(z_0)| + \sum_{k=1}^{n} \log\left|\frac{1-\bar{a}_k z_0}{z_0 - a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{e^{it}+z_0}{e^{it}-z_0} \cdot \log|f(e^{it})|\,dt$$

Idea of the proof Recenter z_0 to 0 using $\mathrm{Aut}\,\Delta$.



$L, M : \Delta \longrightarrow \Delta$

$$L(w) = \frac{w + z_0}{1 + w\bar{z}_0}$$

$$M(w) = \frac{w - z_0}{1 - w\bar{z}_0}$$

Note L, M are inverses & $L(0) = z_0$.

Let $\tilde{f} = f \circ L$. Apply Jensen to $\tilde{f}$.

Claim Zeroes of $\tilde{f}$ in Δ are $M(a_1), \dots M(a_n)$

Proof $\tilde{f}(z) = 0 \iff f\,L(z) = 0 \iff L(z) = a_k$

$\iff z = M\,L(z) = M(a_k)$.

By Jensen for $\tilde{f}$:

$$\log|\tilde{f}(0)| + \sum_{k=1}^{n} \log \frac{1}{M(a_k)} = \frac{1}{2\pi}\int_0^{2\pi} \log|\tilde{f}(e^{is})|\, ds$$

$$\iff \log|f(z_0)| + \sum_{k=1}^{n} \log\left|\frac{1 - a_k \bar{z}_0}{a_k - z_0}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(L(e^{is}))|\, ds \quad (1)$$

We change variables $e^{is} = M(e^{it})$. Then

$$f(L(e^{is})) = f\,L\,M(e^{it}) = f(e^{it}).$$

Furthermore,

$$ds = \text{Poisson Kernel} \cdot dt.$$

This was proven in Lecture 3, Main Claim.

Thus RHS of (1) $= \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})| \cdot \text{Poisson Kernel} \cdot dt.$

With this observation, (1) yields Poisson – Jensen.

§2. Applications of Jensen

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, $f(0) = 1$.

- $M(R) = \sup_{|z|=R} |f(z)|$ = growth of f
- $N(R) = \#$ zeroes of f in $\Delta(0,R)$ with multiplicities

Apply Jensen in $\Delta(0, 3R)$:

$$\underbrace{\log|f(0)|}_{0} + \sum_{|a_k|<3R} \log\left|\frac{3R}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(3Re^{it})|\,dt$$

$$\leq \log M(3R)$$

$$\Rightarrow \log M(3R) \geq \sum_{|a_k|<R} \log\left|\frac{3R}{a_k}\right| + \sum_{R\leq|a_k|<3R} \log\left|\frac{3R}{a_k}\right|$$

$$\geq \sum_{k=1}^{N(R)} \log 3 + \sum_{R\leq|a_k|<3R} \log 1 = N(R)\log 3 > N(R).$$

Conclusion: $N(R) < \log M(3R)$.

What do we learn from this? $\exists$ correlation between

- growth of entire functions $M(R)$
- distribution of their zeroes $N(R)$

The higher the N, the higher the M (at R & $3R$).

Prototypical Example f polynomial, $\deg f = d$

- $N(R) = d$ if $R \gg 0$ by Fundamental Thm Algebra
- $M(R) \sim R^d$.

Thus $$\frac{\log M(R)}{\log R} \longrightarrow d. \quad \text{as } R \longrightarrow \infty$$

The converse is also true. If $\lim\limits_{R \to \infty} \frac{\log M(R)}{\log R} = d \Rightarrow$

$\Rightarrow \log M(R) < (d+1) \log R$ for $R \gg 0$

$\Rightarrow |f(z)| < |z|^{d+1}$ for $|z| \gg 0 \Rightarrow f$ polynomial by

Generalized Liouville. (Math 220A, HWK 4, Problem 3)

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§3. Order of entire functions

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire. We first consider

$$M(R) = \sup_{|z|=R} |f(z)|$$ growth of f

Goal We want to measure growth of entire functions such as

[I] polynomials

[II] e^{z}, e^{z^2}, e^{z^3}, ... ,

[III] $e^{e^{z}}$, $e^{e^{e^{z}}}$, $e^{e^{e^{e^{z}}}}$ ...

Case [I] We have seen $\dfrac{\log M(R)}{\log R} \longrightarrow d$ & conversely.

This quantity is a good measure of growth but only in this case.

Case [II] The examples in [II] roughly speaking "grow like $e^{\text{polynomial}}$". For those, we need one log to get the exponent, and one additional log to use the measure in $\boxed{I}$.

Case $\boxed{III}$ These examples grow very fast, and we will have less to say about them.

Case $\boxed{II}$ motivates the following:

Definition (Conway XI. 2. 15)

Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be entire. The order of f is

$$\lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}.$$

This may be infinite.

Examples

$\boxed{i}$ We have $\lambda(fg) \leq \max(\lambda(f), \lambda(g))$ (HWK 4)

$$\lambda(f+g) \leq \max(\lambda(f), \lambda(g)).$$

$\boxed{ii}$ $f = e^{P}$, $\deg P = d \implies \text{order}(f) = d = \deg P$

(exercise).

$\boxed{iii}$ $f(z) = e^{e^{z}} \implies \text{order}(f) = \infty.$ (exercise)

$\boxed{iv}$ $f(z) = \cos z, \sin z$ have order 1

(HWK 4).

$f(z) = \cos\sqrt{z}$ has order $\frac{1}{2}$