Solutions: Homework 3

Problem 1. Show that if \( u_n : G \to \mathbb{R} \) are subharmonic/superharmonic converging locally uniformly to \( u : G \to \mathbb{R} \), then \( u \) is also subharmonic/superharmonic.

Proof. We prove the statement for subharmonic functions. The same proof works for superharmonic functions, with just the opposite inequality.

Note first that \( u \) is continuous, since continuity is a local property, and \( u_n \to u \) locally uniformly.

Let \( \Delta(a;r) \subset G \). Since the \( u_n \)'s are subharmonic, we have
\[
u_n(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u_n(a + re^{i\theta}) d\theta.
\]

Since the \( u_n \)'s coverge locally uniformly to \( u \), and \( \partial \Delta(a;r) \) is a compact set, it follows that \( u_n \) converges to \( u \) uniformly on \( \partial \Delta(a;r) \). So, we have
\[
\frac{1}{2\pi} \int_0^{2\pi} u_n(a + re^{i\theta}) d\theta \to \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta
\]
as \( n \to \infty \). Since \( u_n(a) \to u(a) \) as \( n \to \infty \), we have
\[
u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta
\]
and hence \( u \) is subharmonic. \( \square \)

Problem 2. Let \( \phi : G \to \mathbb{R} \) be subharmonic, and let \( \overline{\Delta} \subset G \) be a closed disc in \( G \). Consider the Poisson modification \( \tilde{\phi} : G \to \mathbb{R} \) of \( \phi \) along \( \Delta \). Show that \( \tilde{\phi} \) is also subharmonic.

Proof. Let \( a \in G \). If \( a \in G \setminus \Delta \), let \( R \) be chosen so that \( \overline{\Delta}(a,R) \subset G \) and furthermore the mean value inequality is satisfied for \( \phi \) in \( \overline{\Delta}(a,R) \). Then, for all \( 0 \leq r < R \), we have
\[
\tilde{\phi}(a) = \phi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(a + re^{it}) dt \leq \frac{1}{2\pi} \int_0^{2\pi} \tilde{\phi}(a + re^{it}) dt
\]
using that \( \phi \leq \tilde{\phi} \).
For $a \in \Delta$, let $\overline{\Delta}(a, R) \subset \Delta$. For $r < R$, we have

$$\overline{\phi}(a) = h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a + re^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} \overline{\phi}(a + re^{it}) \, dt$$

where $h$ is the harmonic function solving Dirichlet Problem, so that $\phi = h$ in $\Delta$.

The analysis above shows that $\overline{\phi}$ satisfies the mean value inequality in sufficiently small discs, so $\overline{\phi}$ is subharmonic. \hfill \Box

**Problem 3.** Show that the Dirichlet Problem cannot be solved in the punctured disc $G = \Delta(0, 1) \setminus \{0\}$.

*Exhibit a continuous function $f : \partial G \to \mathbb{R}$ which cannot be obtained as the boundary value of a harmonic function in $G$, continuous in $\overline{G}$.*

**Proof.** Note that $\partial G = \partial \Delta \cap \{0\}$. Define $f : \partial G \to \mathbb{R}$ setting

$$f(0) = 1, \quad f(z) = 0 \text{ for } z \in \partial \Delta.$$

Assume that the Dirichlet problem can be solved for $(G, f)$, and let $u$ be the solution. Then $u$ is continuous at 0, so by Problem Set 2, Problem 2, we conclude that $u$ extends to a harmonic function across 0. Thus $u$ is harmonic in $\Delta$. We can then apply the mean value property conclude

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it}) \, dt$$

for all $r < 1$. Since $u$ is continuous in $\overline{G}$, it is uniformly continuous there and thus $u(re^{it}) \to u(e^{it})$ uniformly as $r \to 1$. (This can be seen from the definition. Fix $\epsilon > 0$. We can find $\delta$ such that $|x - y| < \delta$ implies $|u(x) - u(y)| < \epsilon$. Use this for $x = re^{it}, y = e^{it}$.) Making $r \to 1$ in the above, we obtain

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \, dt = 0.$$

This contradicts $u(0) = 1$. Thus the Dirichlet problem cannot be solved for $(G, f)$. \hfill \Box
Problem 4. Let $G$ be open connected, $\zeta_0 \in \partial G$. We say $G$ admits a barrier at $\zeta_0$ provided there exists $\omega : G \to \mathbb{R}$ harmonic, continuous in $\overline{G}$, such that $\omega(\zeta_0) = 0$ and $\omega > 0$ on $\partial G \setminus \{\zeta_0\}$.

Show that if the Dirichlet Problem is solvable in $G$, then $G$ admits a barrier at each $\zeta_0 \in \partial G$.

Proof. Let $f(z) = |z - \zeta_0|$. Since the Dirichlet Problem is solvable in $G$, we can find $\omega$ continuous in $\overline{G}$, harmonic in $G$ such that

$$\omega|_{\partial G} = f.$$ 

In particular $\omega(\zeta_0) = f(\zeta_0) = 0$ and for $z \in \partial G \setminus \{\zeta_0\}$, we have $\omega(z) = |z - \zeta_0| > 0$.

\[\square\]

Problem 5. Let $G$ be a region, $\zeta_0 \in \partial G$, and let $\ell$ be a half-line starting at $\zeta_0$ that intersects $\overline{G}$ only at $\zeta_0$. Let $\zeta_1 \neq \zeta_0$ be a point on the half-line $\ell$. Show that $\zeta_0$ is a barrier for $\partial G$.

Proof. Let

$$\ell = \{t\zeta_0 + (1 - t)\zeta_1 : 0 \leq t \leq 1\}.$$ 

Then $\ell \cap \overline{G} = \{\zeta_0\}$. Let $\omega : \overline{G} \to \mathbb{R}$ be defined by

$$\omega(z) = \text{Re}(\sqrt{\frac{z - \zeta_0}{z - \zeta_1}})$$

where $\sqrt{a} = e^{\frac{1}{2}\text{Log}(a)}$ for any $a \in \mathbb{C} \setminus (-\infty, 0]$ and Log is the principal logarithm, and $\sqrt{0} = 0$.

Note that

$$f(z) = \frac{z - \zeta_0}{z - \zeta_1}$$

is injective and $f(\ell) = (-\infty, 0]$ since

$$f(t\zeta_0 + (1 - t)\zeta_1) = -\frac{1 - t}{t}.$$ 

Therefore,

$$f(G) \subset \mathbb{C} \setminus (-\infty, 0].$$
In consequence, $\sqrt{f}$ can be defined as a holomorphic function on $G$. Then $\omega = \text{Re}(\sqrt{f})$ is harmonic on $G$. Also, $\sqrt{z}$ is continuous on $\mathbb{C} \setminus (-\infty, 0)$ and hence $\omega$ is continuous on $\overline{G}$.

Now, $\omega(\zeta_0) = 0$. Since
$$\sqrt{z} : \mathbb{C} \setminus (-\infty, 0) \to \{z : \text{Re } z > 0\}$$
is a surjective map, we have $\omega(z) > 0$ for all $z \in \partial G \setminus \{\zeta_0\}$. Hence, $\omega$ is a barrier for $\partial G$ at $\zeta_0$.

\[ \square \]

**Problem 6.** Let $\mathcal{H}$ be the family of harmonic functions $h : \Delta \to \mathbb{R}$ with $h(0) = 1$ and $h(z) > 0$ for $z \in \Delta$. Show that every sequence in $\mathcal{H}$ admits a subsequence that converges locally uniformly to a function in $\mathcal{H}$.

**Proof.** Let $\{h_n\}$ be a sequence in $\mathcal{H}$. Write $h_n = \text{Re } f_n$. Thus $\text{Re } f_n > 0$. We know $\text{Re } f_n(0) = 1$. By possibly changing $f_n$ by an imaginary constant, we may furthermore assume
$$f_n(0) = 1.$$  

We already have seen in Math 220B, Homework 4, Problem 2 that $\{f_n\}$ is a normal family. (On the Qualifying Exam, this statement would require a proof since it is based on a homework question.) Passing to a subsequence, we may assume $f_n \to f$ with $f(0) = 1$. Letting $h = \text{Re } f$, we see
$$h_n = \text{Re } f_n \to \text{Re } f = h.$$  

Furthermore, $h$ is harmonic, $h(0) = 1$.

We show $h(z) > 0$ for $z \in \Delta$. Note that
$$h(z) = \lim h_n(z) \geq 0$$
for $z \in \Delta$. If $h(z_0) = 0$ for some $z_0 \in \Delta$, then $z_0$ would be a maximum for $h$, violating the maximum principle unless $h \equiv 0$. However, the latter situation is impossible since $h(0) = 1$. Thus $h > 0$ in $\Delta$, proving that $h \in \mathcal{H}$. \[ \square \]