Solutions: Homework 5

Problem 1. Let \( f, g \) be two entire functions of finite order \( \lambda \). Assume 
\[ f(a_n) = g(a_n) \]
for a sequence \( \{a_n\}_{n \geq 0} \) with
\[ \sum_{n=0}^{\infty} \frac{1}{|a_n|^{\lambda+1}} = \infty. \]
Show that \( f = g \).

Proof. Let us suppose that \( f \neq g \). Let \( h = f - g \). Then
\[ \text{order} (h) \leq \max \{ \text{order} (f), \text{order} (g) \} = \lambda. \]
Suppose \( h \) has a zero of order \( m \) at 0 so that
\[ h(z) = z^m H(z). \]
We have seen in class that multiplication by a polynomial does not affect the order. Thus, \( H \) has order \( \leq \lambda \) as well and \( H(a_n) = 0 \).

Now, let \( \{b_n\}_{n \geq 0} \) be the non-zero zeros of \( H \). Then, \( \{a_n\}_{n \geq 0} \subset \{b_n\}_{n \geq 0} \), and hence
\[ \sum_{n=0}^{\infty} \frac{1}{|b_n|^{\lambda+1}} = \infty. \]
In particular, if \( \alpha \) is the exponent of convergence we must have \( \alpha > \lambda + 1 \).
By part (ii) of Problem 5, HW 4, we have \( \alpha \leq \lambda \). This is a contradiction.
Hence, \( h = 0 \) so that \( f = g \). \( \square \)

Problem 2. (i) Find all entire functions \( f \) of finite order such that \( f(\log n) = n \) for all integers \( n \geq 1 \).
(ii) Give an example of an entire function \( f \) with zeroes only at \( \log n \) for integers \( n \geq 1 \).

Proof. (i) Note that in Problem 1 above, we just used the fact that the order of \( f \) and \( g \) is \( \leq \lambda \), not necessarily equal to \( \lambda \). Suppose that \( f \) is an entire function of finite order such that \( f(\log n) = n \) for all integers \( n \geq 1 \). Let \( \lambda \) denote \( \max \{ \text{order of } f, 1 \} \). Let \( N \geq 2 \) be such that for all \( n \geq N \),
\[ \log n \leq n^{\frac{1}{\lambda+1}} \]
Then
\[ \sum_{n=N}^{\infty} \frac{1}{n} \leq \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\lambda+1}} \]

Applying Problem 1 to \( f \) and \( g(z) = e^z \), we see that \( f(z) = e^z \).

(ii) By Theorem 5.12, the function

\[ f(z) = z \prod_{n=1}^{\infty} E_{n-1} \left( \frac{z}{\log(n+1)} \right) \]

is an entire function with zeros only at \( \log n \) for integers \( n \geq 1 \).

\[ \Box \]

**Problem 3.** If \( f \) is an entire function of order \( \lambda \), show that \( f' \) also has order \( \lambda \).

**Proof.** Let \( M'(R) = \sup \{|f'(z)| : |z| = R\} \), and let \( \lambda' \) denote the order of \( f' \).

Let \( |z| = R \). Then, applying Cauchy’s estimate to \( f \) on \( \Delta(z; 1) \subset \Delta(0, R+1) \), we have

\[ |f'(z)| \leq \sup_{|w-z|=1} |f(w)| \leq M(R+1) \]

and hence

\[ M'(R) \leq M(R+1). \]

Thus

\[ \limsup_{R \to \infty} \frac{\log \log M'(R)}{\log R} \leq \limsup_{R \to \infty} \frac{\log \log M(R+1)}{\log(R+1)} = \limsup_{R \to \infty} \frac{\log \log M(R+1)}{\log(R+1)}. \]

This shows that

\[ \lambda' \leq \lambda. \]

For the opposite inequality, WLOG, we can assume that \( f(0) = 0 \) since else we can work with the function \( f - f(0) \) which has the same order as shown in class. We then have

\[ f(z) = \int_0^1 (f(tz))' \, dt = z \int_0^1 f'(tz) \, dt. \]

Note that

\[ M'(R) = \sup_{|w|=R} |f'(w)| = \sup_{|w| \leq R} |f'(w)|. \]
by the maximum modulus principle. Hence for all $|z| = R$, we have

$$|f'(tz)| \leq M'(R)$$

for $0 \leq t \leq 1$, and thus by the above we conclude

$$|f(z)| \leq RM'(R) \implies M(R) \leq RM'(R).$$

Fix $\epsilon > 0$. Then, taking log, we have

$$\log M(R) \leq \log R + \log M'(R) \leq \log R + R^{\lambda' + \epsilon} \leq R^{\lambda' + 2\epsilon}$$

for $R \gg 0$. Thus

$$\lambda = \lim_{R \to \infty} \frac{\log \log M(R)}{\log R} \leq \lambda' + 2\epsilon.$$

As $\epsilon > 0$ is arbitrary, this shows that

$$\lambda \leq \lambda'.$$

In conclusion, $\lambda = \lambda'$.

\[ \square \]

**Problem 4.** Let $f$ be entire, $|f'(z)| \leq e^{|z|}$ and

$$f(\sqrt{n}) = 0 \quad \text{for all } n \in \mathbb{Z}_{>0}.$$

Show that $f = 0$.

**Proof.** Since $|f'(z)| \leq e^{|z|}$ it follows that $f'$ has order $\lambda' \leq 1$. By the previous question, the order of $f$ must satisfy $\lambda \leq 1$. In particular, the rank

$$p \leq h \leq \lambda \leq 1.$$

By definition of the rank, this means that

$$\sum_n \frac{1}{|n|^{p+1}} = \sum_n \frac{1}{n} < \infty$$

which is clearly a contradiction. Thus $f = 0$.  \[ \square \]
Problem 5. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z - \sin z$. 

(i) Show that $f$ is an odd entire function of order less or equal to 1.

(ii) Using (i), show that $f$ can be represented as a product

$$f(z) = \frac{z^3}{6} \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{a_n^2} \right)$$

where $\{a_n\}$ is a sequence of non-zero complex numbers with

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

Proof. (i) The fact that $f$ is entire and odd is clear. For the order, note that if $|z| = R$, we have

$$|f(z)| \leq |z| + |\sin z| \leq |z| + \left| \frac{1}{2i} (e^{iz} - e^{-iz}) \right| \leq |z| + \frac{1}{2} |e^{iz}| + \frac{1}{2} |e^{-iz}|$$

$$\leq |z| + \frac{1}{2} e^{\text{Re}(iz)} + \frac{1}{2} e^{\text{Re}(-iz)} \leq |z| + \frac{1}{2} e^{|iz|} + \frac{1}{2} e^{|-iz|} = |z| + e^{i|z|} \leq R + e^R.$$

Thus

$$\lambda \leq \limsup_{R \to \infty} \frac{\log \log(R + e^R)}{\log R} = 1.$$

(ii) Since $\lambda \leq 1$ by Hadamard’s theorem, we must have $h \leq \lambda \leq 1$. Thus the rank $p \leq 1$, the and by the definition of the rank, we must have

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty,$$

where $\{a_n\}$ denote the zeroes of $f$ not equal to 0. Since $p + 1 \leq 2$ and $a_n \to \infty$, it follows that $|a_n| > 1$ for $n$ sufficiently large and

$$\frac{1}{|a_n|^2} \leq \frac{1}{|a_n|^{p+1}}.$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty.$$

By Weierstraß factorization we have

$$f(z) = z^m e^{\theta} \prod_{n=1}^{\infty} E_1 \left( -\frac{z}{a_n} \right).$$
Recall that in Weierstraß we can increase the value of $p$ without affecting convergence, so using $p = 1$ is justified. Alternatively, one can split this into two cases $p = 0$ which is simpler, and $p = 1$ which is treated explicitly below.

Since $f$ is odd, the zeroes of $f$ come in pairs $(a_n, -a_n)$. We can combine

$$E_1 \left( \frac{z}{a} \right) E_1 \left( -\frac{z}{a} \right) = \left( 1 - \frac{z}{a} \right) e^{\frac{z}{a}} \left( 1 + \frac{z}{a} \right) e^{-\frac{z}{a}} = 1 - \frac{z^2}{a^2}. $$

Thus, we may write

$$f(z) = z^m e^g \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{a_n^2} \right),$$

after relabelling/discarding some of the zeroes. Combining terms is justified by the local absolute convergence of the product.

Note that $m$ is order of $f$ at 0. Computing the Taylor expansion, we see that

$$f(z) = z - \sin z = \frac{z^3}{6} + \ldots $$

Thus $m = 3$.

The degree $q$ of $g$ satisfies

$$q \leq h \leq \lambda \leq 1.$$ 

Thus $g(z) = az + b$ for some $a, b$. Since $f$ is odd, it follows at once that $e^g$ must be even so

$$e^{g(z)} = e^{g(-z)} \implies e^{az+b} = e^{-az+b} \implies e^{2az} = 1 \implies a = 0.$$ 

Thus $g = b$ must be a constant

$$f(z) = z^3 e^b \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{a_n^2} \right).$$ 

Therefore,

$$\lim_{z \to 0} \frac{f(z)}{z^3} = e^b$$

using the fact that the product converges to an entire (hence continuous) function. However,

$$\lim_{z \to 0} \frac{f(z)}{z^3} = \frac{1}{6}$$
as we see from the Taylor expansion for instance. Thus $e^{b} = \frac{1}{6}$ and

$$f(z) = \frac{z^3}{6} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right).$$

\[\square\]

**Problem 6.** Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function of order $\lambda$. Let

$$\mu = \limsup_{n \to \infty} \frac{n \log n}{-\log |c_n|} > 0.$$

Show that $\lambda = \mu$.

(i) First show that $\lambda \geq \mu$ by showing that for all $\epsilon > 0$ we have $\lambda > \mu - \epsilon$.

(ii) Conversely, show that $\lambda \leq \mu$ by showing that $\lambda < \mu + \epsilon$ for all $\epsilon > 0$.

(iii) Let $a > 0$. Show that the function

$$f(z) = \sum_{n} \frac{z^n}{n^{a_n}}$$

is entire and find its order.

**Proof.** (i) Let $0 < \epsilon < \mu$. By definition,

$$n \log n \geq -(\mu - \epsilon) \log |c_n|$$

for infinitely many $n$. Using Cauchy’s estimate, we have

$$|c_n| \leq \frac{M(R)}{R^n}$$

for all $R > 0$. So we have

$$- \log |c_n| \geq n \log R - \log M(R)$$

and hence,

$$\log M(R) \geq n \log R - \frac{n \log n}{\mu - \epsilon}$$

for infinitely many $n$, and for all $R > 0$. Putting $R_n = (en)^{\frac{1}{\mu - \epsilon}}$, we have

$$\log M(R_n) \geq \frac{n}{\mu - \epsilon} = \frac{R_n^{\mu - \epsilon}}{\mu - \epsilon}$$
Since $R_n \to \infty$, we have
\[
\lambda \geq \mu - \epsilon
\]
Since $0 < \epsilon < \mu$ was arbitrary, we have
\[
\lambda \geq \mu
\]
(ii) Fix $\epsilon > 0$. By definition, there exists $N \geq 1$ such that
\[
n \log n \leq - (\mu + \epsilon) \log |c_n|
\]
for all $n \geq N$, i.e.
\[
|c_n| \leq n^{- \frac{\mu}{\mu + \epsilon}}
\]
for all $n \geq N$. This shows that there exists $C \geq 1$ such that
\[
|c_n| \leq Cn^{- \frac{\mu}{\mu + \epsilon}}
\]
for all $n \geq 1$. Now, for $|z| = R$, we have
\[
\left| \sum_{n=0}^{k} c_n z^n \right| \leq \sum_{n=0}^{k} |c_n||z|^n \leq C \sum_{n=0}^{k} R^n n^{- \frac{n}{\mu + \epsilon}} \leq C \sum_{n=0}^{\infty} R^n n^{- \frac{n}{\mu + \epsilon}}
\]
Letting $k \to \infty$, we have
\[
|f(z)| \leq C \sum_{n=0}^{\infty} R^n n^{- \frac{n}{\mu + \epsilon}}
\]
for all $|z| = R$. So we have
\[
M(R) \leq C \sum_{n=0}^{\infty} R^n n^{- \frac{n}{\mu + \epsilon}}
\]
Now, let
\[
S_1 = \sum_{n \leq (2R)^{\mu + \epsilon}} R^n n^{- \frac{n}{\mu + \epsilon}} \quad \text{and} \quad S_2 = \sum_{n > (2R)^{\mu + \epsilon}} R^n n^{- \frac{n}{\mu + \epsilon}}
\]
We have
\[
S_1 = \sum_{n \leq (2R)^{\mu + \epsilon}} R^n n^{- \frac{n}{\mu + \epsilon}} \leq R^{(2R)^{\mu + \epsilon}} \sum_{n \leq (2R)^{\mu + \epsilon}} n^{- \frac{n}{\mu + \epsilon}} \leq R^{(2R)^{\mu + \epsilon}} \sum_{n=1}^{\infty} n^{- \frac{n}{n + \epsilon}} = AR^{(2R)^{\mu + \epsilon}}
\]
where $A = \sum_{n=1}^{\infty} n^{-\frac{m}{n+e}}$. Similarly, we have

$$S_2 = \sum_{n>(2R)^{\mu+\epsilon}} R^n n^{-\frac{n}{n+e}} \leq \sum_{n>(2R)^{\mu+\epsilon}} R^n (2R)^{n} \leq \sum_{n>(2R)^{\mu+\epsilon}} \left(\frac{1}{2}\right)^n \leq 1$$

Putting this together, we have

$$M(R) \leq C(S_1 + S_2) \leq C(AR^{(2R)^{\mu+\epsilon}} + 1)$$

for all $R > 0$. So we have

$$\lambda \leq \mu + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have $\lambda \leq \mu$, and hence, combining part (i), we have $\lambda = \mu$.

(iii) We have

$$\limsup_{n \to \infty} \left(\frac{1}{n^{an}}\right)^{\frac{1}{n}} = \limsup_{n \to \infty} \frac{1}{n^{an}} = 0$$

and hence, $f$ is a power series with $R = \infty$, and is therefore entire. We have

$$\mu = \limsup_{n \to \infty} \frac{n \log n}{-\log n^{-an}} = \limsup_{n \to \infty} \frac{n \log n}{an \log n} = \frac{1}{\alpha} > 0$$

Thus the order of $f$ is $\frac{1}{\alpha}$. \qed

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