Math 220C - Lecture 1

March 28, 2022

$\boxed{1.}$ Harmonic Functions

Theme:

Harmonic functions share many properties with holomorphic functions

- $\boxed{i}$ mean value property & Integral formulas
- $\boxed{ii}$ maximum modulus principle
- $\boxed{iii}$ convergence theorems

& others $\rightsquigarrow$ HWK 1.

"Cauchy" estimates, Liouville, Open Mapping Thm.

Convention $G \subseteq \mathbb{C}$ open & connected. We will assume this from now on.

Recall

$G \subseteq \mathbb{C}$ open & connected

$u : G \longrightarrow \mathbb{R}$ harmonic iff $u \in C^2$ and

$u_{xx} + u_{yy} = 0$. (Laplace equation).

Recall (Harmonic conjugates, Math 220A, Lecture 1).

If $f : U \longrightarrow \mathbb{C}$ holomorphic $\Rightarrow$ $u = \operatorname{Re} f$ harmonic.

$v = \operatorname{Im} f$ harmonic

u, v are said to be harmonic conjugates provided

$f = u + iv$ is holomorphic.

(so that $u = \operatorname{Re} f$, $v = \operatorname{Im} f$). Note that u, v satisfy

the Cauchy Riemann equations

$$u_x = v_y$$

$$u_y = -v_x.$$

Lemma Let G be simply connected.

Any $u : G \longrightarrow \mathbb{R}$ harmonic admits a harmonic conjugate v.

e.g. $f = u + iv$ = holomorphic, $u = \operatorname{Re} f$.

Proof Let $F = u_x - i u_y$.

Claim F holomorphic

Indeed, F is of class C^1 & satisfies CR equations.

$$(u_x)_x = (-u_y)_y \iff u_{xx} + u_{yy} = 0 \quad \text{true}$$

$$(u_x)_y = -(-u_y)_x \iff u_{xy} = u_{yx}. \quad \text{true}$$

$\Rightarrow$ F holomorphic by Math 220, Lecture 2.

Since G is simply connected, F admits a primitive

$\Rightarrow$ $F = f'$ for f holomorphic, $f = \alpha + i\beta$.

$$f' = \alpha_x + i\beta_x = F = u_x - i u_y$$

$\Rightarrow$ $\alpha_x = u_x$ $\Rightarrow$ $\alpha = u + C.$

$\Rightarrow$ $\beta_x = -u_y = -\alpha_y$ $\Rightarrow$ $\alpha_y = u_y$.

Replacing f by $f-c$, we obtain $u = \operatorname{Re} f$ & $v = \operatorname{Im} f$ is the conjugate of u.

Remark

3. Show that the function $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ given by

$$u(z) = \log |z|$$

is harmonic, but it is not the real part of a holomorphic function in $\mathbb{C} \setminus \{0\}$.

Thus the Lemma above fails for G not simply connected.

Corollary

u harmonic $\Longrightarrow$ u is of class $\mathcal{C}^\infty$.

Proof

Indeed, the statement is local. Let $a \in G$. Let $\overline{\Delta}(a,r) \subseteq G$.

Since $\Delta(a,r)$ simply connected, $u = \operatorname{Re} f$, f holomorphic in $\Delta(a,r)$.

A holomorphic function is ∞-many times complex differentiable & thus ∞-many times real differentiable. (Math 220A, Lecture 1).

$\Rightarrow$ u is $\mathcal{C}^\infty$.

First Properties of Harmonic Functions

[i] mean value property (MVP)

[ii] maximum principle (MP)

[iii] Poisson integral formula

Def $u : G \longrightarrow \mathbb{R}$ continuous satisfies MVP if

$\forall\ a \in G$, $\bar{\Delta}(a, r) \subseteq G$.

$$\underbrace{u(a)}_{\text{value at center}} = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} u(a + re^{it})\, dt}_{\text{average values over the boundary.}}$$

Theorem $u : G \longrightarrow \mathbb{R}$ harmonic $\Rightarrow$ u satisfies MVP.

Proof Let $\overline{\Delta}(a, r) \subseteq \Delta(a, R) \subseteq G$ Write

$u = \operatorname{Re} f$, f holomorphic in $\Delta(a, R)$.

Cauchy Integral Formula gives

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Delta(a,r)} \frac{f(z)}{z-a}\, dz.$$

$z = a + re^{it}$

$dz = rie^{it}dt$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} \cdot rie^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{it})\, dt.$$

Take real part on both sides & conclude.

Maximum Principle

$u : G \longrightarrow \mathbb{R}$, $u \in \mathcal{C}^0(G)$ satisfies MVP. Assume

$\exists\, a \in G$, $u(a) \geq u(z)$ $\forall z \in G$. Then u is constant.

Remark

$\boxed{\text{I}}$ u harmonic $\Rightarrow$ u satisfies maximum principle

$\boxed{\text{II}}$ u harmonic. $\Rightarrow$ $-u$ harmonic

$\Rightarrow$ $-u$ satisfies maximum principle

$\Rightarrow$ u satisfies minimum principle

Georg Friedrich Bernhard Riemann

17 September 1826 – 20 July 1866

Eine harmonische Function u kann nicht in einem Punkt im Innern ein Minimum oder ein Maximum haben, wenn sie nicht uberall constant ist.

(A harmonic function u cannot have either a minimum or a maximum at an interior point unless it is constant.)

"Grundlagen fur eine allgemeine Theorie der Functionen einer veranderlichen complexen Grosse”

Dissertation Gottingen (1851)