Math 220C - Lecture 10

April 18, 2022

§ 1. Jensen's Formula (Conway XI.1)

$f : G \longrightarrow \mathbb{C}$ holomorphic, f nowhere zero in G, $\bar{\Delta}(0,r) \subseteq G$.

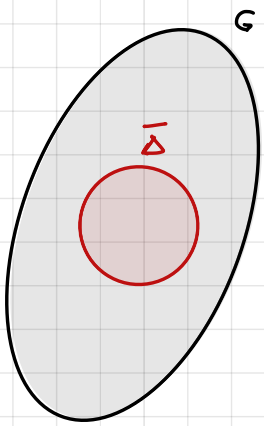
Recall from HWK 1

5. Let $U \subset \mathbb{C}$ be open connected.

(i) Show that if $h : U \to \mathbb{C}$ is holomorphic and nowhere zero in U, then

$$u(z) = \log |h(z)|$$

is harmonic in U.



Mean Value Property for $\log |f|$ gives

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Question What if f has zeroes?

The zeroes of f will give corrections to the formula.

Theorem $f : G \to \mathbb{C}$ holomorphic, $\bar{\Delta}(0,r) \subseteq G.$, $f(0) \neq 0$.

Let $a_1, \dots, a_k$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt.$$

Proof Shrinking G, we may assume $G = \Delta(0,R)$

We may assume $r = 1$. Indeed, otherwise let

$$f^{new}(z) = f(rz) \text{ defined in } G^{new} = \Delta\left(0, \frac{R}{r}\right) \supseteq \bar{\Delta}(0,1).$$

When f is holomorphic in $\Delta(0,R) \supseteq \bar{\Delta}(0,1)$, we show

$$\log |f(0)| - \sum_{k=1}^{n} \log |a_k| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt. \qquad (*)$$

Proof of (*) Let

- $a_1, \ldots, a_k$ be zeroes of f in $\Delta = \Delta(0,1)$
- $b_1, \ldots, b_m$ be zeroes of f on $\partial\Delta$.

Recall $\varphi_a : \bar{\Delta} \longrightarrow \bar{\Delta}$, $\partial\Delta \longrightarrow \partial\Delta$, $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

Let
$$F(z) = f(z) \Big/ \prod_{j=1}^{k} \varphi_{a_j}(z) \cdot \prod_{j=1}^{m} \frac{b_j}{b_j - z}$$

Note that F has no zeroes in $\bar{\Delta}$. & in fact in a neighborhood of $\bar{\Delta}$. Note

$$F(0) = f(0) \Big/ \prod_{j=1}^{m} (-a_j)$$

By the previous observation applied to F

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{it})| \, dt. \quad (1).$$

By substitution, we find

$$\log|F(0)| = \log|f(0)| - \sum_{j=1}^{k} \log|a_k| \qquad (2)$$

$$\int_0^{2\pi} \log|F(e^{it})|\,dt = \int_0^{2\pi} \log|f(e^{it})|\,dt$$

0 (see below)

$$- \sum_{j=1}^{k} \int_0^{2\pi} \log|\varphi_{a_j}(e^{it})|\,dt$$

0 (claim)

$$+ \sum_{j=1}^{m} \int_0^{2\pi} \log\left|\frac{b_j}{b_j - e^{it}}\right| dt$$

$$= \int_0^{2\pi} \log|f(e^{it})|\,dt. \qquad (3)$$

Here we used $\varphi_{a_j} : \partial\Delta \longrightarrow \partial\Delta$ so that

$$|\varphi_{a_j}(e^{it})| = 1 \implies \log|\varphi_{a_j}(e^{it})| = 0.$$

Jensen's formula follows from (1), (2), (3).

Claim $\displaystyle\int_0^{2\pi} \log\left|\frac{b}{b - e^{it}}\right| dt = 0 \quad \forall\, |b| = 1.$

Proof of the claim Let $b = e^{i\alpha}$. Then

$$\int_0^{2\pi} \log\left|\frac{b}{b-e^{it}}\right| dt = \int_0^{2\pi} \log\left|\frac{e^{i\alpha}}{e^{i\alpha}-e^{it}}\right| dt$$

$$= \int_0^{2\pi} \log\left|\frac{1}{1-e^{i(t-\alpha)}}\right| dt \qquad \swarrow \; t \longrightarrow t+\alpha$$

$$= \int_0^{2\pi} \log\frac{1}{|1-e^{it}|}\, dt$$

$$= -\int_0^{2\pi} \log|1-e^{it}|\, dt \overset{?}{=} 0 .$$

We note that

$$|1-e^{it}|^2 = (1-\cos t)^2 + \sin^2 t = 2 - 2\cos t = 4\sin^2\frac{t}{2}.$$

We need to show

$$\int_0^{2\pi} \log\left|2\sin\frac{t}{2}\right| dt = 0 \quad \overset{t=2u}{\Longleftrightarrow}$$

$$\Longleftrightarrow \quad \int_0^{\pi} \log|2\sin u|\, du = 0$$

$$\Longleftrightarrow \quad \int_0^{\pi} \log 2\, du + \int_0^{\pi} \log \sin u\, du = 0$$

$$\Longleftrightarrow \quad \boxed{\int_0^{\pi} \log \sin u\, du = -\pi \log 2.}$$

Calculation $\displaystyle\int_0^{\pi} \log \sin u \, du = -\pi \log 2.$

Convergence

$$\int_0^{\pi} \log \sin u \, du \le \int_0^{\pi} \log u \, du = u \log u - u \Big|_{u=0}^{u=\pi} < \infty.$$

This uses $\lim_{u \to 0} u \log u = 0.$

Evaluation

$$I = \int_0^{\pi} \log \sin u \, du \overset{u=2v}{=}$$

$$= 2 \int_0^{\pi/2} \log \sin 2v \, dv = \qquad \sin 2v = 2 \sin v \cos v.$$

$$= 2 \int_0^{\pi/2} \log 2 \, dv + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \cos v \, dv$$

$$= \pi \log 2 + 2 \int_0^{\pi/2} \log \sin v \, dv + 2 \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} + v\right) dv$$

$$= \pi \log 2 + 2 \int_0^{\pi} \log \sin v \, dv$$

$$= \pi \log 2 + 2 I \implies I = -\pi \log 2.$$

SUR UN NOUVEL ET IMPORTANT THÉORÈME DE LA THÉORIE DES FONCTIONS

PAR

J. L. W. V. JENSEN.

Monsieur le Professeur,

Lors de votre dernier séjour à Copenhague j'ai eu honneur de vous entretenir au sujet d'une intégrale définie appelée, si je ne me trompe, à jouer un rôle dans la théorie des fonctions analytiques. Comme il me parut que cette question vous interéssa vivement, je profiterai de cette occasion — l'envoi des deux petits mémoires[1] destinés à votre Journal — pour vous communiquer le développement détaillé de mon théorème.

Soit $z = re^{\theta i}$ une variable complexe, et α un nombre complexe différent de zéro, on a pour $r < |\alpha|$,

$$l\left(1-\frac{z}{\alpha}\right) = -\sum_{\nu=1}^{\infty}\frac{1}{\nu}\left(\frac{z}{\alpha}\right)^{\nu}$$

où l désigne la valeur principale du logarithme. En prenant les parties réelles des deux membres et en observant que l'on a $\Re(a) = \frac{1}{2}(a+\bar{a})$,[2] on trouve

$$(1) \qquad l\left|1-\frac{z}{\alpha}\right| = -\sum_{\nu=1}^{\infty}\frac{r^{\nu}}{2\nu}\left(\frac{e^{\nu\theta i}}{\alpha^{\nu}}+\frac{e^{-\nu\theta i}}{\bar{\alpha}^{\nu}}\right), \qquad r=|z|<|\alpha|.$$

[1] (1) *Sur les fonctions entières.*
(2) *Note sur une condition nécessaire et suffisante pour que tous les zéros d'une fonction entière soient réels.*

[2] Ici et dans la suite je désigne toujours par $\Re(a)$ la partie réelle et par $\bar{a}$ la valeur conjuguée de a.

Acta Math 1899, volume 22

Johan Jensen (1859–1925) was a Danish mathematician. He pursued mathematics while worked as a telephone engineer.

Jensen found his formula while unsuccessfully trying to prove the Riemann hypothesis.

He is also known for Jensen's inequality (about convex functions).

Outcome

$$\log|f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,dt.$$

Question How about values not at the center?

§2. Poisson – Jensen formula

We generalize both

– Jensen's formula & Poisson's formula

Theorem Let $f: G \to \mathbb{C}$ holomorphic, $\bar{\Delta}(0,r) \subseteq G$, $z_0 \in \bar{\Delta}(0,r)$. $f(z_0) \neq 0$. Let $a_1, \dots, a_n$ be the zeroes of f in $\Delta(0,r)$. Then

$$\log |f(z_0)| + \underbrace{\sum_{k=1}^{n} \log \left| \frac{r^2 - \bar{a}_k z_0}{r(z_0 - a_k)} \right|}_{\text{contribution from zeroes}} = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\operatorname{Re} \frac{re^{it} + z_0}{re^{it} - z_0}}_{\substack{\text{Poisson Kernel} \\ \text{(Lectures 3 \& 4)}}} \cdot \log |f(re^{it})| \, dt$$

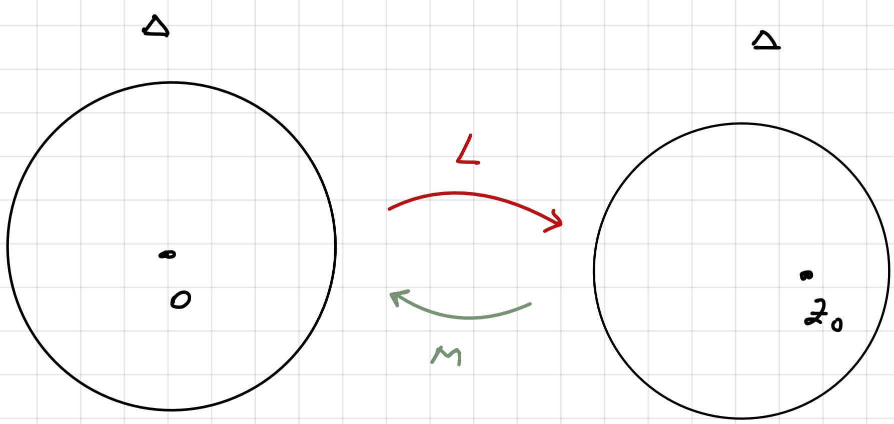
Remark $\boxed{I}$ When $z_0 = 0$, we recover Jensen's formula.

$\boxed{II}$ If f has no zeroes, this becomes Poisson's formula for the function $\log |f|$, which is harmonic in this case.

Proof WLOG $r=1$. Let $\Delta = \Delta(0,1)$. **WTS**

$$\log|f(z_0)| + \sum_{k=1}^{n} \log\left|\frac{1-\bar{a}_k z_0}{z_0 - a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{e^{it}+z_0}{e^{it}-z_0}\cdot \log|f(e^{it})|\,dt$$

Idea of the proof Recenter z_0 to 0 using $\mathrm{Aut}\,\Delta$.



$L, M : \Delta \longrightarrow \Delta$

$$L(w) = \frac{w + z_0}{1 + w\bar{z}_0}$$

$$M(w) = \frac{w - z_0}{1 - w\bar{z}_0}$$

Note L, M are inverses & $L(0) = w_0$.

Let $\tilde{f} = f \circ L$. Apply Jensen to $\tilde{f}$.

Claim Zeroes of $\tilde{f}$ in Δ are $M(a_1), \dots M(a_n)$

Proof $\tilde{f}(z) = 0 \iff f L(z) = 0 \iff L(z) = a_k$

$\iff z = ML(z) = M(a_k)$.

By Jensen for $\tilde{f}$:

$$\log|\tilde{f}(0)| + \sum_{k=1}^{n} \log \frac{1}{M(a_k)} = \frac{1}{2\pi}\int_0^{2\pi} \log|\tilde{f}(e^{is})|\, ds$$

$$\Longleftrightarrow \quad \log|f(z_0)| + \sum_{k=1}^{n} \log\left|\frac{1-a_k\bar{z}_0}{a_k - z_0}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(L(e^{is}))|\, ds \quad (1)$$

We change variables $e^{is} = M(e^{it})$. Then

$$f(L(e^{is})) = f\,L\,M(e^{it}) = f(e^{it}).$$

Furthermore,

$$ds = \text{Poisson Kernel} \cdot dt.$$

This was proven in Lecture 3, Main Claim.

Thus RHS of (1) $= \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})| \cdot \text{Poisson Kernel} \cdot dt.$

With this observation, (1) yields Poisson – Jensen.