Math 220C - Lecture 11

April 20, 2022

§1. Applications of Jensen

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire, $f(0) = 1$.

- $M(R) = \sup_{|z|=R} |f(z)|$ = growth of f
- $N(R)$ = # zeroes of f in $\Delta(0,R)$ with multiplicities

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Apply Jensen in $\Delta(0, 3R)$:

$$\underbrace{\log |f(0)|}_{0} + \sum_{|a_k| < 3R} \log \left| \frac{3R}{a_k} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(3Re^{it})| \, dt$$

$$\leq \log M(3R)$$

$$\Rightarrow \log M(3R) \geq \sum_{|a_k| < R} \log \left| \frac{3R}{a_k} \right| + \sum_{R \leq |a_k| < 3R} \log \left| \frac{3R}{a_k} \right|$$

$$\geq \sum_{k=1}^{N(R)} \log 3 + \sum_{R \leq |a_k| < 3R} \log 1 = N(R) \log 3 > N(R).$$

Conclusion $\boxed{N(R) < \log M(3R).}$

What do we learn from this? $\exists$ correlation between

- growth of entire functions $M(R)$
- distribution of their zeroes $N(R)$

The higher the N, the higher the M (at R & $3R$).

Prototypical Example f polynomial, $\deg f = d$

- $N(R) = d$ if $R \gg 0$ by Fundamental Thm Algebra
- $M(R) \sim R^d$.

Thus $$\frac{\log M(R)}{\log R} \longrightarrow d. \quad \text{as } R \longrightarrow \infty$$

The converse is also true. If $\lim\limits_{R\to\infty} \dfrac{\log M(R)}{\log R} = d \Rightarrow$

$\Rightarrow \log M(R) < (d+1) \log R$ for $R \gg 0$

$\Rightarrow |f(z)| < |z|^{d+1}$ for $|z| \gg 0 \Rightarrow f$ polynomial by

Generalized Liouville. (Math 220A, HWK 7, Problem IV. 3. 1.)

Main Question

$f: \mathbb{C} \longrightarrow \mathbb{C}$ entire function

Establish relationship between

$\{$ Growth of $f \}$ $\longleftrightarrow$ $\{$ Distributions of zeros $\}$

Subquestion: How do we interpret the two sides mathematically?

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§ 2. Left hand side – Order of entire functions

$f: \mathbb{C} \longrightarrow \mathbb{C}$ entire. We first consider

$$M(R) = \sup_{|z|=R} |f(z)|$$ growth of f

Goal We want to measure growth of entire functions

such as

[I] polynomials

[II] e^{z}, e^{z^2}, e^{z^3}, ...,

[III] $e^{e^{z}}$, $e^{e^{z^2}}$, $e^{e^{e^{z^2}}}$ ...

Case [I] We have seen $\dfrac{\log M(R)}{\log R} \longrightarrow d$ & conversely.

This quantity is a good measure of growth but only in this case.

Case [II] The examples in [II] roughly speaking "grow like $e^{\text{polynomial}}$". For those, we need one log to get the exponent, and one additional log to use the measure in [I].

Case [III] These examples grow very fast, and we will have less to say about them.

Case $\boxed{2c}$ motivates the following:

Definition (Conway XI. 2. 15)

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be entire. The order of f is

$$\lambda = \limsup_{R \to \infty} \frac{\log \log M(R)}{\log R}.$$

This may be infinite.

Intuitively, " $f(z) \sim e^{|z|^{\lambda}}$ ".

Question How to prove a function f has order λ?

We need to show two statements:

[I] $\forall\, \varepsilon > 0\ \exists\, r$ such that $|f(z)| < e^{|z|^{\lambda+\varepsilon}}$ $\forall\, |z| > r$

This shows $M(R) < e^{R^{\lambda+\varepsilon}}$ $\forall\, R > r$ &

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \le \lambda + \varepsilon \quad \forall\, \varepsilon. \quad \overset{\varepsilon\to 0}{\Longrightarrow} \quad \lambda(f) \le \lambda$$

[II] $\forall\, \varepsilon > 0 \quad \exists\, z_n \longrightarrow \infty$ with $|f(z_n)| > e^{|z_n|^{\lambda-\varepsilon}}$

This shows

$$\lambda(f) = \limsup_{R\to\infty} \frac{\log\log M(R)}{\log R} \ge \limsup_{n\to\infty} \frac{\log\log|f(z_n)|}{\log|z_n|} \ge \lambda - \varepsilon$$

$$\overset{\varepsilon\to 0}{\Longrightarrow} \quad \lambda(f) \ge \lambda.$$

Examples

$\boxed{i}$ $\lambda(z^m) = 0$, $M(R) = R^m \Rightarrow \lambda = 0$.

$\boxed{ii}$ $f = e^P$, $\deg P = d \Rightarrow \text{order}(f) = d = \deg P$

(exercise).

$\boxed{iii}$ $f(z) = e^{e^z} \Rightarrow \text{order}(f) = \infty$. (exercise)

$\boxed{iv}$ $f(z) = \cos z, \sin z$ have order 1

(HWK 5)

$f(z) = \cos\sqrt{z}$ has order $\frac{1}{2}$

$\boxed{v}$ We have $\lambda(fg) \leq \max(\lambda(f), \lambda(g))$ (HWK 4)

$\lambda(f+g) \leq \max(\lambda(f), \lambda(g))$.

§3. Right hand side & Distribution (growth) of zeroes

Assume f has zeroes at

$$|a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots \,.\quad a_n \longrightarrow \infty \,,\quad a_n \neq 0$$

Several quantities attached to growth of zeroes:

$\boxed{\text{i}}$ rank $= p$

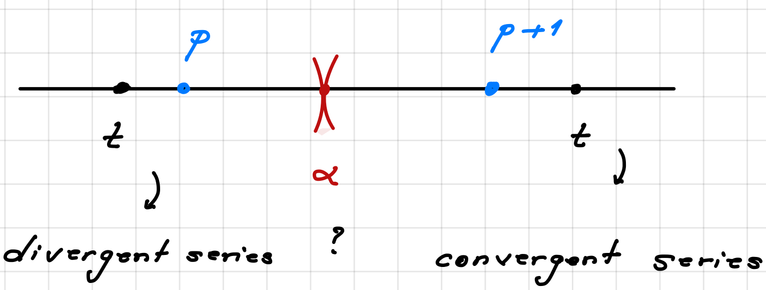
The smallest integer p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$.

If such a p doesn't exist, $p = \infty$.

$\boxed{\text{ii}}$ critical exponent (HWK 4, #5)

$\alpha = \inf \left\{ t > 0 : \sum \frac{1}{|a_n|^t} < \infty \right\}$ may not be an integer

By the homework



Thus by definition $p \leq \alpha \leq p+1$.

If $\alpha \notin \mathbb{Z}$ then α determines p uniquely.

[iii] $N(R)$ = # zeroes in $\Delta(0,R)$ with multiplicity

Fact* (we will not use/prove)

$$\alpha = \limsup_{R\to\infty} \frac{\log N(R)}{\log R}$$

Example* Let $a_n = n^3$, $n > 0$. Then

$$N(R) = \#\{n : n^3 < R\} \sim R^{1/3} \Rightarrow \frac{\log N(R)}{\log R} \to \frac{1}{3}$$

Note

$$\sum \frac{1}{n^{3t}} < \infty \iff 3t > 1 \iff t > \frac{1}{3} \quad \text{so } \alpha = \frac{1}{3}.$$

harmonic series

Upshot We have defined the following quantities measuring growth / distribution of zeroes

$N(R)$, α, p.

Note $N(R)$ determines α, α determines p if $\alpha \notin \mathbb{Z}$.

Best for us: p (or h to be defined next).