Math 220C - Lecture 12

April 22, 2022

§ 1. Distribution (growth) of zeroes

Assume f has zeroes at

$$|a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots \quad . \quad a_n \longrightarrow \infty \, , \quad a_n \neq 0$$

Several quantities attached to growth of zeroes:

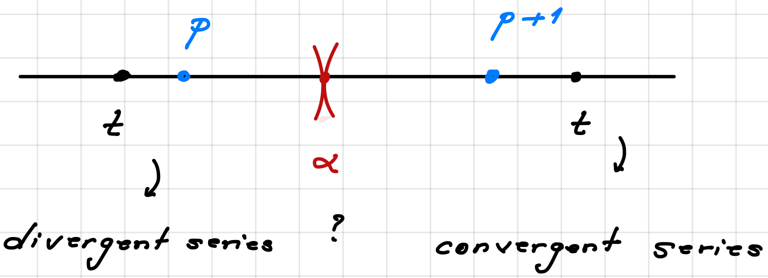
$\boxed{i}$ rank $= p$

The smallest integer p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$.

If such a p doesn't exist, $p = \infty$.

$\boxed{ii}$ critical exponent (HWK 4, #5)

$\alpha = \inf \left\{ t > 0 : \sum \frac{1}{|a_n|^t} < \infty \right\}$ may not be an integer



Thus by definition $p \le \alpha \le p+1$.

If $\alpha \notin \mathbb{Z}$ then α determines p uniquely.

[iii] $N(R)$ = # zeroes in $\Delta(0,R)$ with multiplicity

Fact* (we will not use/prove)

$$\alpha = \limsup_{R\to\infty} \frac{\log N(R)}{\log R}$$

Example* Let $a_n = n^3$, $n>0$. then

$$N(R) = \#\{n : n^3 < R\} \sim R^{1/3} \Rightarrow \frac{\log N(R)}{\log R} \to \frac{1}{3}$$

Note

$$\sum \frac{1}{n^{3t}} < \infty \iff 3t > 1 \iff t > \frac{1}{3} \quad \text{so} \quad \alpha = \frac{1}{3}.$$

harmonic series

Upshot We have defined the following quantities measuring growth/distribution of zeroes

$N(R)$, α, p.

Note $N(R)$ determines α, α determines p if $\alpha \notin \mathbb{Z}$.

Best for us: p (or h to be defined next).

Small variation — Genus of an entire function

Let f has zeroes at $a_1, a_2 \dots, a_n, \dots, a_k \neq 0$.

where $\{a_n\}$ has rank p. $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$

Recall Weierstraß Factorization

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right).$$

Recall

$$E_p(z) = \begin{cases} 1 - z, & p = 0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), & p > 0 \end{cases}$$

Define

$$h = \text{genus}(f) = \begin{cases} \max(p, q) & \text{if } g \text{ polynomial of degree } q \\ \infty & \text{if } g \text{ not polynomial or } p = \infty. \end{cases}$$

If the exponential e^g doesn't appear then $h = p$.

In general $p \leq h$.

Example (Math 220B)

$$\boxed{\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)}$$ factorization of sine.

Rewrite this as

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi}$$

$$= z \prod_{n=1}^{\infty} E_1\left(\frac{z}{n\pi}\right) E_1\left(-\frac{z}{n\pi}\right)$$

$\Rightarrow g$ doesn't appear. Thus genus $h = p$.

The zeroes are at $n\pi$, $n \in \mathbb{Z}$. We want

$$\sum_{n \neq 0} \frac{1}{|n\pi|^{p+1}} < \infty \iff p+1 > 1 \iff p > 0.$$ Thus the

harmonic series

smallest p equals 1.

$$\boxed{\text{The genus of } z \longrightarrow \sin z \text{ equals } 1.}$$

§2 Revisiting the Main Question (now made precise)

Establish relationship between

$$\{\text{Growth of } f\} \longleftrightarrow \{\text{Growth of zeroes}\}$$

measured by λ — measured by h = genus.

Answer **Theorem** (Hadamard)

$$h \le \lambda \le h+1$$

Remarks $\boxed{i}$ If $\lambda \notin \mathbb{Z}$ then λ determines h uniquely.

$\boxed{ii}$ If e^{g} doesn't appear then $h = p$ so in this case.

$$p \le \lambda \le p+1$$

$\boxed{iii}$ We have $p \le h \le \lambda$ so the order bounds the p in the Weierstraß Factorization. The statement that we can take $p \le \lambda$ is called Hadamard Factorization.

Remarks $\boxed{IV}$ The theorem doesn't assume h, λ finite.

If one of them is infinite $\Rightarrow$ so is the other.

$\boxed{V}$ These ideas played an important role in Hadamard's proof of Prime Number Theorem. (1896)

Conclusion $\exists$ connections between

- $M(R)$ and λ by definition $\lambda = \limsup_{R \to \infty} \dfrac{\log\log M(R)}{\log R}$
- $N(R)$, α, p as we saw above
- λ and $h = \max(p, q)$ via Hadamard $h \leq \lambda \leq h+1$

Jacques Hadamard (1865 – 1963)

Proved the Prime Number Theorem.

Advisor: Émile Picard.

Students: Maurice Fréchet, André Weil

Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann [1];

PAR M. J. HADAMARD.

1. La décomposition d'une fonction entière $F(x)$ en facteurs primaires, d'après la méthode de M. Weierstrass,

$$F(x) = e^{G(x)} \prod_{p=1}^{\infty} \left(1 - \frac{x}{\xi_p}\right) e^{Q_p(x)} \tag{1}$$

a conduit à la notion du genre de la fonction F.

On dit que F est du genre E si, dans le second membre de l'équation (1), tous les polynômes Q_p sont de degré E, et que la fonction entière $G(x)$ se réduise également à un polynôme de degré E au plus.

Dans un article inséré au *Bulletin de la Société mathématique de France* [2], M. Poincaré a démontré une propriété des fonctions de genre E. L'énoncé auquel il est parvenu est le suivant :

Dans une fonction entière de genre E, *le coefficient de x^m, mul-*

[1] Les principaux résultats contenus dans le présent Mémoire ont été présentés à l'Académie des Sciences dans un travail couronné en 1892 (grand prix des Sciences mathématiques).

[2] Année 1883, pages 136 et suiv.

Journal de Mathematiques Pures et Appliquées

(1893)

§ 3 Applications – Picard's Theorems (weak versions)

To illustrate the power of this result we show:

Application A (Conway 3.6)

f entire & not constant & finite order

$\Longrightarrow$ f omits at most one value.

Remark Little Picard removes the assumption the order is finite.

Proof Assume f omits $\alpha \neq \beta$. Define

$$f^{new} = \frac{f - \alpha}{\beta - \alpha} \quad \text{omits } 0 \ \& \ 1.$$

Since f^{new} omits 0 $\Rightarrow$ $f^{new} = e^{g}$ & f^{new} omits 1

$\Rightarrow$ g omits 0. Since order (f^{new}) = order $(f) < \infty$

$\Longrightarrow$ genus of f^{new} is finite by Hadamard. $\Rightarrow$ g polynomial.

& g omits 0. $\rightarrow$ g = constant $\Rightarrow$ f constant. False!

Easy Observations (used above)

$\boxed{i}$ $\lambda \geq 0$

We have seen $|f(z)| \leq e^{|z|^{\lambda+\varepsilon}}$ if $|z| \geq R_\varepsilon$ last lecture.

If $\lambda < 0$, let $\varepsilon > 0$ with $\lambda + \varepsilon < 0$. Then $|f(z)| \leq e^{|z|^0} = e$

for $|z| \geq R$ and $|f(z)| \leq M$ for $|z| \leq R$ by continuity. Thus

f bounded $\Rightarrow$ f constant (order 0). Thus $\lambda \geq 0$

$\boxed{ii}$ f & αf have the same order $\forall \alpha \neq 0$

HWK

Indeed $\lambda(\alpha f) \leq \max(\lambda(\alpha), \lambda(f)) = \max(0, \lambda(f)) = \lambda(f)$ by $\boxed{i}$

the previous line

Similarly $\lambda(f) = \lambda\left(\alpha f \cdot \frac{1}{\alpha}\right) \leq \lambda(\alpha f)$. Thus $\lambda(f) = \lambda(\alpha f)$.

$\boxed{iii}$ f & $f - \alpha$ have the same order

Same proof as in $\boxed{ii}$ using sums versus products

$\boxed{iv}$ f & Pf have the same order $\forall$ P polynomial.

We have $f \leq Pf$ if $|z| >> 0$ (if $\deg P > 0$.) $\Rightarrow \lambda(f) \leq \lambda(Pf)$.

Also $\lambda(Pf) \leq \max(\lambda(P), \lambda(f)) = \max(0, \lambda(f)) = \lambda(f)$. (HWK 4) $\boxed{i}$

Thus $\lambda(Pf) = \lambda(f)$

Application B

f entire of finite order & $\lambda \notin \mathbb{Z}$ $\Rightarrow$ f assumes each of its values infinitely many times.

Remark Great Picard strengthens this result.

Proof Let α be a value of f. Define $f^{new} = f - \alpha$. We show f^{new} has ∞-many zeroes. Assume f^{new} has finitely many zeroes $a_1, \dots, a_n$. Let $P = \prod_{k=1}^{n} (z - a_k)$. Then f^{new}/P has no zeroes so it equals e^g. $\Rightarrow$

$\Rightarrow f^{new} = P e^g$. Note by previous remarks we have

order f = order f^{new} = order $e^g < \infty$. $\Rightarrow$ genus $< \infty$

$\Rightarrow g$ polynomial & order $(e^g) = \deg g \in \mathbb{Z}$. $\Rightarrow$ order $(f) \in \mathbb{Z}$ contradiction.