Math 220C - Lecture 13

April 25, 2022

§0. Last time Conway XI. 3.

$f : \mathbb{C} \longrightarrow \mathbb{C}$ entire of order λ, $f \not\equiv 0$.

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right), \quad p = \text{rank}(f).$$

genus $h = \max(p, \deg g)$ if g polynomial or ∞ otherwise.

Hadamard's theorem (1893)

$$h \leq \lambda \leq h+1$$

Plan for the Proof of Hadamard $h \leq \lambda \leq h+1$

[I] $\lambda \leq h+1$ (today).

[II] $h \leq \lambda$ — $p \leq \lambda$ (today).

$\deg g \leq \lambda$ (next time)

Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann [1];

PAR M. J. HADAMARD.

1. La décomposition d'une fonction entière $F(x)$ en facteurs primaires, d'après la méthode de M. Weierstrass,

$$F(x) = e^{G(x)} \prod_{p=1}^{\infty} \left(1 - \frac{x}{\xi_p}\right) e^{Q_p(x)} \tag{1}$$

a conduit à la notion du genre de la fonction F.

On dit que F est du genre E si, dans le second membre de l'équation (1), tous les polynômes Q_p sont de degré E, et que la fonction entière $G(x)$ se réduise également à un polynôme de degré E au plus.

Dans un article inséré au *Bulletin de la Société mathématique de France* [2], M. Poincaré a démontré une propriété des fonctions de genre E. L'énoncé auquel il est parvenu est le suivant :

Dans une fonction entière de genre E, *le coefficient de* x^m, *mul-*

(1) Les principaux résultats contenus dans le présent Mémoire ont été présentés à l'Académie des Sciences dans un travail couronné en 1892 (grand prix des Sciences mathématiques).

(2) Année 1883, pages 136 et suiv.

Journal de Mathematiques Pures et Appliquées

(1893)

§ 1. First half of Hadamard

WTS $\lambda \leq h+1$ $(*)$

WLOG h finite, else we're done.

Key Lemma $\log |E_p(w)| \leq C_p |w|^{p+1}$ for some $C_p > 0$.

Proof of $(*)$ WTS $\lambda \leq h+1$.

Recall $f(z) = z^m e^g \prod_n E_p\left(\frac{z}{a_n}\right)$

Recall order $(uv) \leq \max(\text{order } u, \text{order } v)$.

Recall order $(z^m) = 0 \leq h+1$

order $(e^g) = \deg g \leq h < h+1$.

We show order $\prod_n E_p\left(\frac{z}{a_n}\right) \leq p+1 \leq h+1$.

Note

$$\log\left|\prod_n E_p\left(\frac{z}{a_n}\right)\right| = \sum_n \log\left|E_p\left(\frac{z}{a_n}\right)\right|$$

Lemma $\Big\{$

$$\leq C_p \sum_n \left|\frac{z}{a_n}\right|^{p+1} = K\,|z|^{p+1}$$

where $K = C_p \sum \frac{1}{|a_n|^{p+1}} < \infty$.

Thus order $\leq p+1$, as needed.

Remark (will not prove/use)

order $\prod_n E_p\left(\frac{z}{a_n}\right) = \alpha$ (exercise in Conway).

Proof of Lemma

Recall $E_p(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \dots + \frac{w^p}{p}\right)$.

We induct on p.

When $p = 0$,

$$\log|1-w| \leq \log(1+|w|) \leq |w| \quad \text{so take } C_0 = 2.$$

Inductive step

$\boxed{i}$ When $|w| \geq \frac{1}{2}$: Note

$$E_p(w) = E_{p-1}(w)\exp\left(\frac{w^p}{p}\right)$$

$$\Longrightarrow \log|E_p(w)| = \log|E_{p-1}(w)| + \log\left|\exp\left(\frac{w^p}{p}\right)\right|$$

$$\leq c_{p-1}|w|^p + \log\exp\operatorname{Re}\left(\frac{w^p}{p}\right)$$

$$= c_{p-1}|w|^p + \operatorname{Re}\left(\frac{w^p}{p}\right)$$

$$\leq c_{p-1}|w|^p + \left|\frac{w^p}{p}\right| = \left(c_{p-1} + \frac{1}{p}\right)|w|^p$$

$$\leq 2\left(c_{p-1} + \frac{1}{p}\right)|w|^{p+1} \quad \text{since } |w| \geq \frac{1}{2}.$$

[ii] When $|w| \le \frac{1}{2}$. Note

$$E_p(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^p}{p}\right)$$

$$= \exp\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)$$

using Taylor expansion

$$\log(1-w) = -w - \frac{w^2}{2} - \cdots - \frac{w^k}{k} - \cdots \quad \text{for } |w| < 1.$$

Then

$$\log|E_p(w)| = \log\left|\exp\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)\right|$$

$$= \mathrm{Re}\left(-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right)$$

$$\le \left|-\frac{w^{p+1}}{p+1} - \frac{w^{p+2}}{p+2} - \cdots\right|$$

$$\le \sum_{k \ge p+1} \left|\frac{w^k}{k}\right| = |w|^{p+1} \sum_{k \ge 0} \frac{|w|^k}{p+k+1}$$

$$\le |w|^{p+1} \sum_{k \ge 0} |w|^k \le$$

$$\le |w|^{p+1} \sum_{k \ge 0} \left(\frac{1}{2}\right)^k = 2|w|^{p+1}.$$

Take $c_p = \max\left(2,\ 2\left(c_{p-1} + \frac{1}{p}\right)\right)$. We obtain in both cases

$$\log |E_p(w)| \leq c_p |w|^{p+1}$$ as needed.

§ 2. Second half of Hadamard

We show $h \leq \lambda$.

WLOG λ finite & $f(0) = 1$

Indeed, write $f(z) = c z^m \tilde{f}(z)$ with $\tilde{f}(0) = 1$. Note

$$\text{order } \tilde{f} = \text{order } f \quad \& \quad \text{genus } \tilde{f} = \text{genus } f.$$

WTS If λ is finite, then

$\boxed{i}$ $p \leq \lambda$

$\boxed{ii}$ g polynomial of degree $\leq \lambda$.

where $f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$.

Proof of $\boxed{i}$

By HWK 4, Problem 5:

$\alpha \leq \lambda$ and by Lecture 10, $\rho \leq \alpha$. Thus $\rho \leq \lambda$.

Preparations for the proof of $\boxed{ii}$

Let $m \leq \lambda < m+1$.

Write
$$f(z) = e^{g(z)} P(z) \quad \text{where } P(z) = \prod_n E_p\left(\frac{z}{a_n}\right)$$

We will prove

$$D^{m+1} g = 0 \quad \text{in } \mathbb{C} \setminus \{a_1, a_2 \dots a_n \dots\}.$$

This will show $D^{m+1} g = 0$ in $\mathbb{C}$, say by identity principle. $\Rightarrow$ g polynomial of degree $\leq m$.

Here D = derivative $= \frac{\partial}{\partial z}$

Aside — Logarithmic derivative

Issue: Taking derivatives of products is messy. It is easier to take logarithmic derivatives

$$h \text{ holomorphic} \Rightarrow \frac{h'}{h} = \text{logarithmic derivative}$$

$$= \text{holomorphic away from } \mathrm{Zero}(h)$$

Addition formula

$$h = fg \implies \frac{h'}{h} = \frac{f'}{f} + \frac{g'}{g}.$$

$$h' = f'g + fg' \Rightarrow \frac{h'}{h} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

Inductively $h = f_1 \cdots f_s \implies \dfrac{h'}{h} = \dfrac{f_1'}{f_1} + \cdots + \dfrac{f_s'}{f_s}$

We prove the same for infinite products.

Proposition Let $f_k : \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic & assume

$$h = \prod_{k=1}^{\infty} f_k$$ converges absolutely & locally uniformly.

Away from $\mathrm{Zero}(h)$, we have $$\frac{h'}{h} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

The RHS converges locally uniformly on $\mathcal{U} \setminus \mathrm{Zero}(h)$.

This follows from applying the finite case to the partial products.

Key: $u_n \overset{l.u.}{\rightrightarrows} u$ then $\dfrac{u_n'}{u_n} \overset{l.u.}{\rightrightarrows} \dfrac{u'}{u}$ away from $\mathrm{Zero}(u)$.

Indeed, by Weierstraß convergence thm $\Rightarrow$ $u_n' \overset{l.u}{\rightrightarrows} u'$.

This gives $\dfrac{u_n'}{u_n} \overset{l.u.}{\rightrightarrows} \dfrac{u'}{u}$ away from $\mathrm{Zero}(u)$ using two basic results:

[i] $u_n \overset{l.u}{\rightrightarrows} u$ then $\dfrac{1}{u_n} \overset{l.u.}{\rightrightarrows} \dfrac{1}{u}$ away from $\mathrm{Zero}(u)$

[ii] $u_n \overset{l.u.}{\rightrightarrows} u$ and $v_n \overset{l.u.}{\rightrightarrows} v$ then $u_n v_n \overset{l.u.}{\rightrightarrows} uv$

Back to Hadamard – Take logarithmic derivatives

$$f = e^{g} P \implies \frac{f'}{f} = g' + \frac{P'}{P}$$

Take m usual derivatives next to get

$$D^m \frac{f'}{f} = D^{m+1} g + D^m \frac{P'}{P}.$$

We will show $\boxed{D^m \frac{f'}{f} = D^m \frac{P'}{P}} \implies \boxed{D^{m+1} g = 0}$ as needed.