Math 220C - Lecture 16

May 2, 2022

Minicourse on Riemann Surfaces

First Goal — Introduction & basic properties

- So far, we have done complex analysis for domains $G \subseteq \mathbb{C}$ & studied holomorphic functions

- Many results carry over if we replace $G \subseteq \mathbb{C}$ by Riemann surfaces.

- The subject merges ideas from Complex Analysis with Geometry & Topology

- Connections w/ many fields

topology

differential geometry

algebraic geometry

arithmetic geometry

number theory

dynamics

....

Historically, Riemann Surfaces arose from attempts to understand analytic continuation of multi-valued functions

e.g. log ; algebraic functions

See Conway $\overline{\underline{IX}}$.

Riemann Surfaces – first defined by Riemann in his dissertation 1851

– the same dissertation considered the Riemann Mapping Theorem (Math 220B).

Bernhard Riemann (1826 – 1866).

Riemann surfaces were introduced by Riemann in his dissertation at Göttingen (1851). This transformed complex analysis, merging it with topology & algebraic geometry.

"We restrict the variables x, y to a finite domain by considering as the locus of the point O no longer the plane A but a surface T spread over the plane"

"We admit the possibility ... of covering the same part of the plane several times. However in such a case, we assume that those parts of the surface lying on top of one another are not connected by a line. Thus a fold or a splitting of parts of the surface cannot occur."

Translation by. R. Remmert,

"From Riemann surfaces to complex spaces"

Soc. Math. France, Congr 3 (1998)

- Klein : "Riemann's methods were regarded almost with distrust by other mathematicians".

— Ahlfors : "Riemann's writings are full of almost cryptic messages to the future".

§1. Sheaves

Sheaves in agriculture – a collection of stalks bundled together

Sheaves in mathematics

– we seek to formalize the concept of "function-like objects" e.g. holomorphic functions on Riemann surfaces

– the most elegant way of doing so is via sheaf theory

Definition Let X be a topological space. A *presheaf* of sets, abelian groups, rings ... is an assignment

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U})$$

of sets, abelian groups, rings... for all $\mathcal{U} \subseteq X$ open.

& restriction maps

$$\rho_{\mathcal{U}V} : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(V) \quad \text{when } V \subseteq \mathcal{U}$$

which should be homomorphisms of ... . We require

[i] $\rho_{\mathcal{U}\mathcal{U}} : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(\mathcal{U})$ is the identity

[ii] $\forall\ W \subseteq V \subseteq \mathcal{U}$ we have

$$\rho_{\mathcal{U}W} = \rho_{VW} \circ \rho_{\mathcal{U}V} : \begin{tikzcd} \mathcal{F}(\mathcal{U}) \arrow[r, "\rho_{\mathcal{U}V}"'] \arrow[rr, bend right, "\rho_{\mathcal{U}W}"'] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"'] & \mathcal{F}(W) \end{tikzcd}$$

Terminology

[i] elements $s \in \mathcal{F}(\mathcal{U})$ are called sections.

[ii] restriction maps $\rho_{\mathcal{U}V}(s) = s|_V$.

Definition A presheaf $\mathcal{F} \to X$ is a **sheaf** provided

$\forall$ $\mathcal{U} = \bigcup_i \mathcal{U}_i$ open cover, $s_i \in \mathcal{F}(\mathcal{U}_i)$ with

$$s_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$$

$\Rightarrow$ $\exists!$ $s \in \mathcal{F}(\mathcal{U})$ such that $s|_{\mathcal{U}_i} = s_i$.

Examples

[I] X topological space, $\mathcal{F} = \mathcal{C}$ is the sheaf:

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ continuous} \}.$$

with the usual restriction maps $\mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(V)$, $f \to f|_V$.

[II] $X \subseteq \mathbb{R}^n$ open, $\mathcal{F} = \mathcal{C}^k$, $0 \le k \le \infty$, $k = \omega$

$$\mathcal{U} \rightsquigarrow \mathcal{F}(\mathcal{U}) = \{ f : \mathcal{U} \longrightarrow \mathbb{C} \text{ of class } \mathcal{C}^k \}$$

is a sheaf.

III $G \subseteq \mathbb{C}$ open, $\mathcal{F} = \mathcal{O}_G$, $U \subseteq G$ open

$$\mathcal{O}_G(U) = \{f : U \to \mathbb{C} \text{ holomorphic}\}$$ is a sheaf.

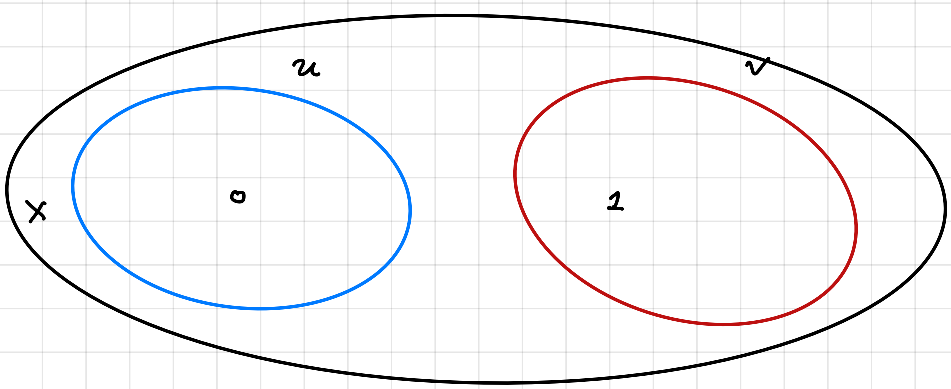
IV $p \in X$ topological space. The skyscraper sheaf

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

V The constant presheaf over X = top. space

$$\underline{\mathbb{C}}(U) := \{f : U \to \mathbb{C} \text{ constant}\}$$ is not a sheaf.

Why? Assume $U, V \subseteq X$



$U \cap V = \emptyset$

$f_1 \equiv 0$ on U

$f_2 \equiv 1$ on V

$\Rightarrow f_1|_{U \cap V} = f_2|_{U \cap V}$.

Let $W = U \cup V$. Gluing fails.

However

$\underline{\mathbb{C}}^{sh}: \mathcal{U} \longrightarrow \{ f: \mathcal{U} \longrightarrow \mathbb{C} \text{ locally constant} \}$ is a sheaf.

VII Restriction of sheaves to open sets

$\mathcal{F} \longrightarrow X$ sheaf, $\mathcal{U} \subseteq X$ open

Define $\mathcal{F}/_{\mathcal{U}}$ a sheaf over $\mathcal{U}$ via

$\mathcal{F}/_{\mathcal{U}}(V) = \mathcal{F}(V)$ for $V \subseteq \mathcal{U}$ open. Note that

$V \subseteq X$ is also open since $\mathcal{U} \subseteq X$ is open, so the above makes sense.

Sheaves were discovered by Leray in the 40s as POW.

His papers were sent to Hopf in Zürich for publication.

Stalks & Germs

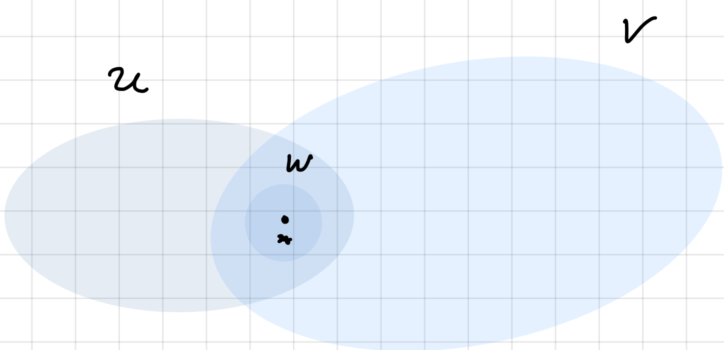
$\mathcal{F} \longrightarrow X$ presheaf, $x \in X$

Consider pairs (U, s), consisting of $x \in U \subseteq X$ open and

$s \in \mathcal{F}(U)$ a section.

$\boxed{(U, s) \sim (V, t)}$ provided $\exists\ x \in W \subseteq U \cap V$ open with

$s|_W = t|_W$.

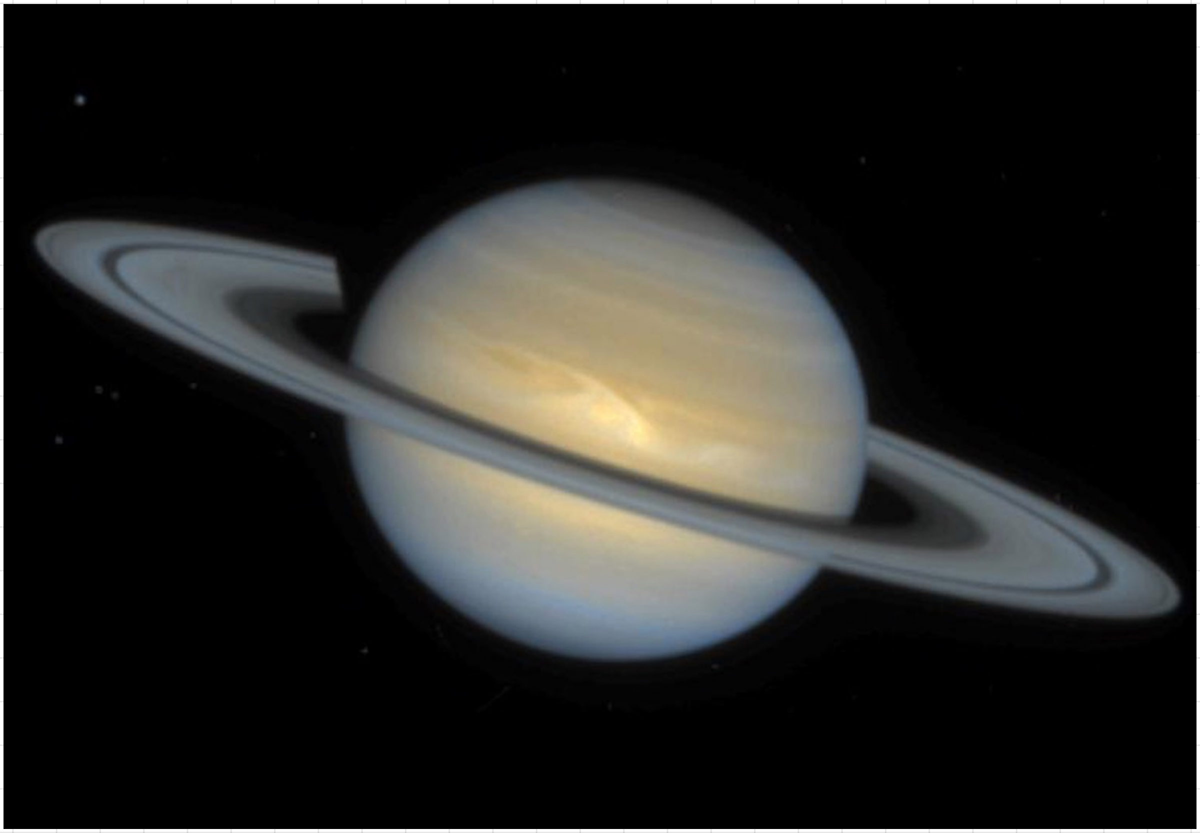


This is an equivalence relation.

The stalk of $\mathcal{F}_x$ is the set of equivalence classes.

An equivalence class is called a germ.

We have $\boxed{\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)}$



Ringed spaces

A ringed space $(X, \mathcal{O}_X)$ is the datum of

$\boxed{i}$ X topological space

$\boxed{ii}$ sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras of complex valued continuous functions. ("regular functions")

Morphisms

$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces

$\boxed{i}$ f continuous

$\boxed{ii}$ $\forall\, U \subseteq Y$, $\varphi \in \mathcal{O}_Y(U)$. the pullback $\varphi \circ f : f^{-1}(U) \longrightarrow \mathbb{C}$ is a section of $\mathcal{O}_X(f^{-1}U)$.

Remark By $\boxed{i}$, $f^{-1}U$ is open which is needed for $\boxed{ii}$ to make sense.

Example $G, G' \subseteq \mathbb{C}$

$f : (G, \mathcal{O}_G) \longrightarrow (G', \mathcal{O}_{G'})$ is a morphism of ringed spaces

$\Longleftrightarrow$ f holomorphic.

Why? $\Leftarrow$ If φ holomorphic in $U \subseteq G'$ & f holomorphic then $\varphi \circ f$ is holomorphic in $f^{-1}(U)$.

$\Rightarrow$ If f morphism, let $\varphi(z) = z$ holomorphic in $U = G'$

Then $\varphi \circ f = f$ is holomorphic by condition $\boxed{ii}$.

Remark We have the notion of an isomorphism.

Remark If X ringed space, $(X, \mathcal{O}_X)$.

$U \subseteq X$ open $\Rightarrow (U, \mathcal{O}_X|_U)$ is a ringed space.

Aside (Point Set Topology) X Hausdorff

$\boxed{\text{i}}$ X is 2^{nd} countable if X admits a countable base for its topology

$\boxed{\text{ii}}$ X is paracompact if all open covers admit a locally finite subcover

$\boxed{\text{iii}}$ $X = \bigcup \mathcal{U}_\alpha$ open cover. A partition of unity $f_\alpha : X \longrightarrow \mathbb{R}$ continuous satisfies

- $\operatorname{supp} f_\alpha \subseteq \mathcal{U}_\alpha$ & $\operatorname{supp} f_\alpha$ is locally finite
- $\sum f_\alpha = 1$, $0 \le f_\alpha \le 1$.

In general $\boxed{\text{ii}} \iff \boxed{\text{iii}}$. $\boxed{\text{i}} \iff \boxed{\text{ii}}$ for manifolds.

Definition A $\mathcal{C}^k$-manifold ($k \geq 0$, $k = \infty$, $k = \omega$) of dim. n.

[i] X Hausdorff, connected, 2^{nd} countable

[ii] $\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ and open subsets $G_\alpha \subseteq \mathbb{R}^n$ such that $(\mathcal{U}_\alpha, \mathcal{O}_X|_{\mathcal{U}_\alpha})$ is isomorphic as a ringed space to $(G_\alpha, \mathcal{C}^k)$.

Definition A Riemann surface $(X, \mathcal{O}_X)$ is

[i] X Hausdorff, connected, 2^{nd} countable top space

[ii] $\exists$ open cover $X = \bigcup \mathcal{U}_\alpha$ and open subsets $G_\alpha \subseteq \mathbb{C}$ such that $(\mathcal{U}_\alpha, \mathcal{O}_X|_{\mathcal{U}_\alpha})$ is isomorphic$_{\mathbb{C}}$ as a ringed space to $(G_\alpha, \mathcal{O}_{G_\alpha})$.

Holomorphic functions

Let X be a Riemann surface & $U \subseteq X$ open.

A holomorphic function on U is a section of $\mathcal{O}_X(U)$.

Holomorphic maps between Riemann Surfaces

$f : X \longrightarrow Y$ holomorphic iff f is a morphism of ringed spaces.