Math 220C - Lecture 18

May 6, 2022

Last time We defined Riemann Surfaces & coordinate charts.

Conversely

Let X be a topological space (Hausdorff, 2^{nd} countable)

$X = \bigcup_\alpha U_\alpha$ open cover. Assume we are given

- $\varphi_\alpha : U_\alpha \longrightarrow G_\alpha$ homeomorphisms, $G_\alpha \subseteq \mathbb{C}$ such that

- $\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha (U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta (U_\alpha \cap U_\beta)$ biholomorphism

These are called compatible coordinate charts

Then X becomes a Riemann surface.

Issue Define the sheaf $\mathcal{O}_X$.

Note $\quad U$ open $\iff U \cap U_\alpha$ open $\iff \varphi_\alpha (U \cap U_\alpha)$ open in G_α.

Declare $f : U \longrightarrow \mathbb{C}$ to be a section of $\mathcal{O}_X$ provided.

$$f \in \mathcal{O}_X (U) \iff f \circ \varphi_\alpha^{-1} \text{ holomorphic in } \varphi_\alpha (U \cap U_\alpha). \ \forall \alpha .$$

Check $\mathcal{O}_X$ is a sheaf & $(X, \mathcal{O}_X)$ is a Riemann Surface.

Examples of Riemann surfaces

$\boxed{I}$ not compact

$\boxed{II}$ compact

Non-compact examples

$\boxed{a}$ $G \subseteq \mathbb{C}$ open subset is a Riemann surface

$\boxed{b}$ $X \subseteq \mathbb{C}^2$, $X = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\} \subseteq \mathbb{C}^2$.

Assume $\forall\, p \in X$,

$$f_x(p) \neq 0 \quad \text{or} \quad f_y(p) \neq 0.$$

Claim X is a Riemann surface

Proof We construct charts & show they are compatible.

Let $p \in X$.

- if $f_y(p) \neq 0 \implies$ by implicit function theorem,

$\exists\ p \in \mathcal{U} \subseteq X$ open such that

$y = g(x)$ for $(x,y) \in \mathcal{U}$ where $g : V \longrightarrow \mathbb{C}$ is holomorphic.

Then $U \longrightarrow G$, $(x,y) \longrightarrow x$ has inverse

$x \longrightarrow (x, g(x))$. $\Longrightarrow$ U is a chart

- If $f_x(p) \neq 0$, we similarly have

$x = h(y)$ for $(x,y) \in U$, $h : H \longrightarrow \mathbb{C}$ holomorphic

Then $U \longrightarrow H$ is a chart $(x,y) \longrightarrow y$ with inverse

$y \longrightarrow (h(y), y)$. $\Longrightarrow$ U is a chart

Compatibility Charts of the first type are clearly compatible. Same for charts of 2^{nd} type.

We check compatibility between charts of different types.

WLOG we may assume we are around a point p with

$f_x(p) \neq 0$ & $f_y(p) \neq 0$.

Then the change of coordinates is

$$
\begin{tikzcd}
 & \mathcal{U} \arrow[r] \arrow[dl] & G \arrow[dll, "T"'] \\
H & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (x,y) \arrow[r] \arrow[dl] & x \arrow[dll, "T"] \\
y & &
\end{tikzcd}
$$

$$T : G \longrightarrow H$$
$$x \longrightarrow y = g(x)$$
$$T^{-1} : H \longrightarrow G$$
$$y \longrightarrow x = h(y)$$

Both T & T^{-1} are holomorphic, as needed.

Compact Riemann Surfaces

$\boxed{a}$ $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

We construct charts $\mathcal{U}_0, \mathcal{U}_\infty$, $X = \mathcal{U}_0 \cup \mathcal{U}_\infty$

$$\mathcal{U}_0 = \{ z : z \neq \infty \} \xrightarrow{\phi_0} \mathbb{C} \quad , \quad z \longrightarrow z$$

$$\mathcal{U}_\infty = \{ z : z \neq 0 \} \xrightarrow{\phi_\infty} \mathbb{C} \quad , \quad z \longrightarrow \frac{1}{z}$$

These two charts are compatible. The transition map is

$$T = \phi_\infty \phi_0^{-1} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \ , \quad z \longrightarrow 1/z \quad \text{biholomorphic}$$

$\Rightarrow X$ Riemann surf

Projective curves

Let $\mathbb{P}^2 = \{ [x:y:z] \,,\ (x,y,z) \neq (0,0,0) \,,\ x,y,z \in \mathbb{C} \} /_{\sim}$

$(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ if $\lambda \in \mathbb{C}^{\times}$

$$U_1 = \{ x \neq 0 \} \xrightarrow{\phi_1} \mathbb{C}^2 \qquad [x:y:z] \to \left(\frac{y}{x}, \frac{z}{x} \right).$$

$$U_2 = \{ y \neq 0 \} \xrightarrow{\phi_2} \mathbb{C}^2 \qquad [x:y:z] \to \left(\frac{x}{y}, \frac{z}{y} \right).$$

$$U_3 = \{ z \neq 0 \} \xrightarrow{\phi_3} \mathbb{C}^2 \qquad [x:y:z] \to \left(\frac{x}{z}, \frac{y}{z} \right).$$

Let f homogeneous of degree d. in variables x, y, z.

(*) if $p \in \mathbb{P}^2$, $f(p) = 0$ then $f_x(p) \neq 0$ or $f_y(p) \neq 0$ or $f_z(p) \neq 0$.

Then

$$X = \{ [x:y:z] : f(x,y,z) = 0 \} \hookrightarrow \mathbb{P}^2$$

is a Riemann Surface (check).

$\boxed{c}$ torus : $\omega_1, \omega_2 \neq 0$, $\omega_1/\omega_2 \notin \mathbb{R}$

Let. $\Lambda = \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \} \hookrightarrow \mathbb{C}$

Let $X = \mathbb{C}/\Lambda$ where Λ acts on $\mathbb{C}$ by translations

Let $\pi : \mathbb{C} \longrightarrow X$

$\boxed{i}$ X has the quotient topology, $\mathcal{U} \subseteq X$:

$\mathcal{U}$ open $\Longleftrightarrow$ $\pi^{-1}\mathcal{U}$ open

$\boxed{ii}$ π continuous & open

$\mathcal{U}$ open, $\pi(\mathcal{U})$ open since $\pi^{-1}\pi\,\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{U} + \lambda$ = open.

$\boxed{iii}$ coordinate charts. Let $\varepsilon < \frac{1}{2} \min_{\lambda \in \Lambda \setminus \{0\}} |\lambda|$.

Let $x \in X$, $\pi(z) = x$, $z \in \mathbb{C}$

Let $\pi : \Delta(z, \varepsilon) \longrightarrow \bar{D}_{z,\varepsilon} = \pi(\Delta(z, \varepsilon))$.

Claim $D_{z,\varepsilon}$ is a chart

π surjective, injective, continuous, open hence a homeomorphism

Claim The charts $D_{z,\varepsilon}$ are compatible

$\varphi_1 : D_{z_1,\varepsilon} \longrightarrow \Delta(z_1, \varepsilon)$

$\varphi_2 : D_{z_2,\varepsilon} \longrightarrow \Delta(z_2, \varepsilon)$

$U = D_{z_1,\varepsilon} \cap D_{z_2,\varepsilon}$

$T = \varphi_2 \varphi_1^{-1}$ is given by $z \longrightarrow z + \lambda$. on $\varphi_1(U)$.

Indeed $\pi T(z) = \pi \varphi_2 \varphi_1^{-1}(z) = \pi \varphi_2 \pi(z) = \pi(z)$

$\Rightarrow T(z) = z + \lambda$. biholomorphic

Conclusion Give X the complex structure determined by these charts. X is a Riemann surface.