Math 220C - Lecture 19

May 11, 2022

§ 1. Basic Results

[a] Identity Theorem

$f, g : X \longrightarrow Y$ holomorphic maps between Riemann Surf.

$S = \{ x : f(x) = g(x) \}$ has a limit point in X.

Then $f \equiv g$.

[b] Open Mapping theorem

$f : X \longrightarrow Y$ holomorphic, non constant $\Rightarrow f$ is open

[c] Maximum Modulus

$f : X \longrightarrow \mathbb{C}$ holomorphic & $|f|$ has a maximum at $p \in X$

$\Rightarrow f$ constant.

Corollary

$f : X \longrightarrow \mathbb{C}$, X compact $\Rightarrow f$ constant

Proof of Open Mapping Theorem

Let $\mathcal{U} \subseteq \mathbb{C}$ be open. We may consider $f\big|_{\mathcal{U}} : \mathcal{U} \longrightarrow Y$ holomorphic & not constant (because of the identity theorem). This way, it suffices to prove the theorem when $\mathcal{U} = X$. Thus we show $f(X)$ is open in Y.

Let $X = \bigcup_\alpha \mathcal{U}_\alpha$, $Y = \bigcup_\alpha \mathcal{U}_\alpha'$. where $\mathcal{U}_\alpha, \mathcal{U}_\alpha'$ are coordinate charts. We may shrink $\mathcal{U}_\alpha$ to assume $f(\mathcal{U}_\alpha) \subseteq \mathcal{U}_\alpha'$! Let $\phi_\alpha : \mathcal{U}_\alpha \xrightarrow{\sim} G_\alpha$, $\phi_\alpha' : \mathcal{U}_\alpha' \xrightarrow{\sim} G_\alpha'$ where $G_\alpha \subseteq \mathbb{C}$, $G_\alpha' \subseteq \mathbb{C}$. Then

$\phi_\alpha' f \phi_\alpha^{-1} : G_\alpha \longrightarrow G_\alpha'$ is holomorphic & not constant (else, we'd have $f =$ constant on $\mathcal{U}_\alpha$ and it would contradict identity principle). By OMT from usual complex analysis, $\phi_\alpha' f \phi_\alpha^{-1}$ is open $\Rightarrow$ f is open since $\phi_\alpha, \phi_\alpha'$ are homeomorphisms. $\Rightarrow$ $f(\mathcal{U}_\alpha)$ is open in $\mathcal{U}_\alpha'$ hence in Y $\Rightarrow$ $f(X) = \bigcup_\alpha f(\mathcal{U}_\alpha) =$ open in Y.

Proof of Maximum Principle

Let $p \in U_\alpha$. Let $\varphi_\alpha : U_\alpha \longrightarrow G_\alpha$ be a chart. Let

$f \circ \varphi_\alpha^{-1} : G_\alpha \longrightarrow \mathbb{C}$, $G_\alpha \subseteq \mathbb{C}$. Then $|f \circ \varphi_\alpha^{-1}|$ has a maximum

at $\varphi_\alpha(p)$. By the usual maximum principle for G_α

$\Rightarrow f \circ \varphi_\alpha^{-1} = \text{constant}$ in G_α $\Rightarrow f = \text{constant}$ in U_α $\Rightarrow$

$\Rightarrow f = \text{constant}$ by the identity theorem.

Rephrasing in terms of sheaves

- $\mathcal{F} \longrightarrow X$, set $H^0(X, \mathcal{F}) := \mathcal{F}(X)$

- X compact $\Rightarrow H^0(X, \mathcal{O}_X) = \mathbb{C}$.

Proof of Identity Principle

$$\Omega = \{ x \in X, \ f = g \text{ in a neighborhood of } x \}$$

Claims

$\boxed{i}$ $\Omega \neq \emptyset$

$\boxed{ii}$ Ω open $\Rightarrow \Omega = X \Rightarrow f \equiv g.$

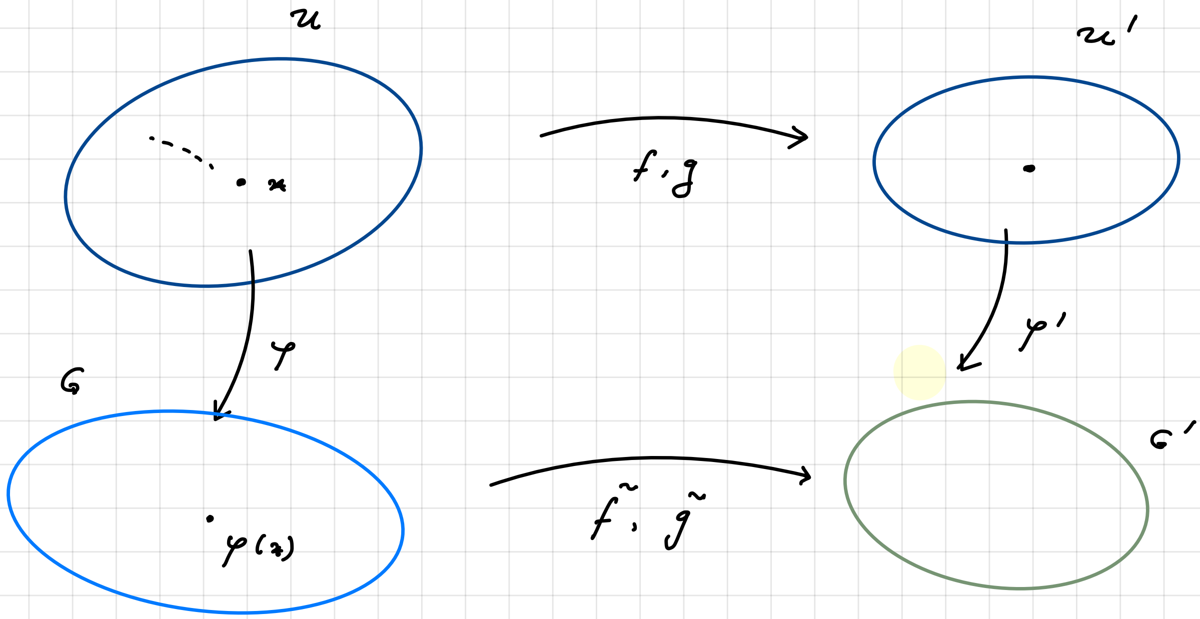
$\boxed{iii}$ Ω closed

Proof of $\boxed{i}$

Let $S = \{ s : f(s) = g(s) \} \subseteq X$ have a limit point x. We show $x \in \Omega$.

Let U be a chart near x, U' a chart in Y near $f(x) = g(x) = y$. Shrinking if needed we may assume

$$f(U) \subseteq U', \quad g(U) \subseteq U', \quad U \text{ connected.}$$



Let $\varphi : \mathcal{U} \longrightarrow G$, $\varphi' : \mathcal{U}' \longrightarrow G'$ be coordinate charts,

$G, G' \subseteq \mathbb{C}$. Let $\tilde f = \varphi' f \varphi^{-1}$, $\tilde g = \varphi' g \varphi^{-1}$. Let

$\tilde S = \{ \tilde s \in G : \tilde f(\tilde s) = \tilde g(\tilde s) \}$. Note $\tilde S \supseteq \varphi(S)$ so $\tilde S$ has

a limit point $\varphi(x) \Rightarrow \tilde f \equiv \tilde g$ in G $\Rightarrow f = g$ in $\mathcal{U}$ $\Rightarrow$

$\Rightarrow x \in \Omega \Rightarrow \Omega \neq \emptyset$.

Part $\boxed{II}$ is clear by definition. Part $\boxed{III}$ is a repetition of the above argument. (check it!)

§2. Questions about functions on Riemann surfaces

Question Is every divisor $D = \sum_{p \in X} n_p [p]$ the divisor of a meromorphic function?

Answer depends on X.

$\boxed{i}$ non-compact $X \subseteq \mathbb{C}$ open

If $D \geq 0$, $n_p \geq 0$ $\forall p$, the question is equivalent to the Weierstraß Problem.

In general write $D = D_+ - D_-$, D_+, D_- effective.

Write $D_+ = \operatorname{div} f_+$, $D_- = \operatorname{div} f_-$, $f = f_+/f_-$. Then

$$D = \operatorname{div} f_+ - \operatorname{div} f_- = \operatorname{div} f_+/f_- = \operatorname{div} f.$$

$\boxed{\text{II}}$ compact X

<u>Example</u> $X = \hat{\mathbb{C}}$. We need $\deg D = 0$ since we already noted $\deg \operatorname{div} f = 0$.

Conversely if $\deg D = 0$, $D = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

If $a_i, b_i \in \mathbb{C}$, let
$$f = \frac{\prod_{i=1}^{n} (z - a_i)}{\prod_{i=1}^{n} (z - b_i)} \Rightarrow \operatorname{div} f = D.$$

If one of the a_i's or b_i's equals ∞, use first a FLT to reduce to the previous case. Thus

$$D \text{ principal} \iff \deg D = 0 \quad \text{if } X = \hat{\mathbb{C}}.$$

Question Given

- $z_1, \dots z_n \in X$, $p_1 \dots p_m \in X$
- $\mu_1 \dots \mu_n \geq 0$, $\nu_1, \dots \nu_m \geq 0$ integers

Want f meromorphic in X

- f has zeroes at z_i of order $\geq \mu_i$
- f has poles at p_i of order $\leq \nu_i$.

Other zeroes are allowed, but no other poles.

Let $D = -\sum_i \mu_i [z_i] + \sum_i \nu_i [p_i]$

Want $\operatorname{div} f - \sum_i \mu_i [z_i] + \sum_i \nu_i [p_i] \geq 0$

$\iff$ $D + \operatorname{div} f \geq 0$ (non-negative coefficients).