Math 220C - Lecture 2

March 30, 2022

Last time

Mean value Property

$$\forall\, a \in G,\ \bar{\Delta}(a,r) \subseteq G,\ u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt.$$

Maximum Principle

$u : G \longrightarrow \mathbb{R}$, $u \in C^0(G)$ satisfies MVP. Assume

$\exists\, a \in G$, $u(a) \geq u(z)$ $\forall\, z \in G$. Then u is constant.

Proof Let $\Omega = \{ z : u(z) = u(a) \} \subseteq G$.

(1) $\Omega \neq \emptyset$ because $a \in \Omega$.

(2) Ω is closed, since u is continuous.

(3) Ω is open.

Then G connected $\Rightarrow \Omega = G \Rightarrow u$ constant.

Proof of (3)

Let $z_0 \in \Omega$. Let $\bar{\Delta}(z_0, r) \subseteq G$. We show $\Delta(z_0, r) \subseteq \Omega$.

Let $w \in \Delta(z_0, r)$. $\Rightarrow \rho = |w - z_0|$. Write MVP. for $\partial\Delta(z_0, \rho)$

$$u(a) = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{it})\, dt.$$

$$\Longleftrightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} \left(u(z_0 + \rho e^{it}) - u(a) \right) dt = 0 \quad \Rightarrow$$

Let $f(t) = u(a) - u(z_0 + \rho e^{it})$. By assumption, $f(t) \geq 0$

since a is a maximum for u.

Using the Lemma, we have $f \equiv 0$. Since $|w - z_0| = \rho$, write

$$w = z_0 + \rho e^{it_0} \Rightarrow f(t_0) = u(a) - u(w) = 0 \Rightarrow u(a) = u(w)$$

$\Rightarrow w \in \Omega \Rightarrow \Delta(z_0, r) \subseteq \Omega \Rightarrow \Omega$ open.

Lemma $f : [0, 2\pi] \longrightarrow \mathbb{R}$, $f \geq 0$ and f continuous

$$\int_0^{2\pi} f(t)\, dt = 0 \Rightarrow f \equiv 0.$$

Proof If $f(t_0) > 0$, by continuity we can find $\delta > 0$

such that $f(t) > \dfrac{f(t_0)}{2}$ $\forall\, t \in (t_0 - \delta, t_0 + \delta) \cap [0, 2\pi]$.

Assume $t_0 \neq 0, 2\pi$ since the proof is similar in those cases.

Then $f \geq 0$ gives

$$0 = \int_0^{2\pi} f(t)\, dt \geq \int_{t_0 - \delta}^{t_0 + \delta} f(t)\, dt > \int_{t_0 - \delta}^{t_0 + \delta} \frac{f(t_0)}{2}\, dt = \delta f(t_0) > 0.$$

contradiction. Thus $f \equiv 0$.

Notation $\partial_\infty G$ = extended boundary in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$$\partial_\infty G = \begin{cases} \partial G, & G \text{ bounded} \\ \partial G \cup \{\infty\}, & G \text{ unbounded} \end{cases}$$

A stronger version (MP^+)

(1) $u : G \longrightarrow \mathbb{R}$, u satisfies MVP in G, u continuous

(2) $\forall\, a \in \partial_\infty G$: $\displaystyle\limsup_{z \to a} u(z) \leq 0$.

Then either $u < 0$ in G or $u \equiv 0$ in G.

Proof We will show $u \leq 0$ in G. By the usual MP, u cannot have a maximum in G unless u = constant. This gives the statement we seek. Indeed, if $\exists\, \alpha \in G$ with $u(\alpha) = 0 \Rightarrow \alpha$ maximum in $G \Rightarrow u \equiv 0$. Else $u(\alpha) < 0$, $\forall\, \alpha \in G$

Thus $u \equiv 0$ or $u < 0$ in G.

To show $u \leq 0$, assume that $\exists\, \alpha \in G$ with $u(\alpha) > 0$.

Let $\varepsilon = u(\alpha) > 0$.

Let $K = \{ z \in G : u(z) \geq \varepsilon \}$. Since $\alpha \in K \Rightarrow K \neq \emptyset$.

Claim K is compact.

Assuming this, u cont., u will achieve a maximum in K at z_0. In particular $u(z_0) \geq \varepsilon$. Outside of K, $u < \varepsilon$. Thus z_0 will achieve a maximum for in u in G. This shows u constant

Condition (2) ensures $u = \text{constant} \leq 0$.

Proof of claim Let $z_n \in K$. We show that passing to a subseq. z_n converges in K. Note $z_n \in \hat{\mathbb{C}}$ & $\hat{\mathbb{C}}$ is compact. Thus WLOG we may assume $z_n \longrightarrow z \in \hat{\mathbb{C}}$ after passing to a subsequence.

Note $u(z_n) \geq \varepsilon$. If $z \in G \Rightarrow u(z) = \lim u(z_n) \geq \varepsilon$. $\Rightarrow z \in K$. as needed. Else $z \in \partial_\infty G$. Then

$$\limsup_{z_n \to z} u(z_n) \geq \varepsilon \quad \text{which contradicts (2).}$$

Thus K is compact.

Corollary G bounded, $u : \overline{G} \longrightarrow \mathbb{R}$ cont., MVP,

$u \equiv 0$ on ∂G $\Rightarrow$ $u \equiv 0$ in G.

Proof We use MP^+. We need to verify condition (2).

G bounded, $\partial_\infty G = \partial G$. If $a \in \partial G$, $\lim\limits_{z \to a} u(z) = \underbrace{u(a)}_{\text{continuity in } \overline{G}} = 0$.

Thus $u < 0$ in G or $u \equiv 0$ in G.

Argue in the same way for $-u$. $\Rightarrow$ either $-u < 0$ in G or $-u \equiv 0$ in G. Thus $u \equiv 0$ in G.

Remark $u, v : \overline{G} \longrightarrow \mathbb{R}$ continuous & harmonic in G.

& G bounded. If

$$u\big|_{\partial G} = v\big|_{\partial G} \Rightarrow u = v \text{ in } G.$$

Thus $u\big|_{\partial G} \rightsquigarrow u$ in G uniquely.

§2. Poisson Formula & Dirichlet Problem

Question 1 $u : \overline{G} \longrightarrow \mathbb{R}$ continuous, harmonic in G, G bounded.

$$u\big/_{\partial G} \rightsquigarrow u \quad \text{uniquely in } G.$$

Find a formula for u in G, from the values $u\big/_{\partial G}$.

We will solve this for $G = \Delta(0,1)$. or $\Delta(a, R)$. $\rightsquigarrow$ Poisson Formula

Question 2 Given $f : \partial G \longrightarrow \mathbb{R}$ continuous, is there

$u : \overline{G} \longrightarrow \mathbb{R}$ continuous and

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \\ u\big/_{\partial G} = f \end{cases}$$

Dirichlet Problem

(boundary value problem)

Harmonic Functions on the unit disc $\Delta = \Delta(0,1)$

Given $u : \overline{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ,

find a formula for $u(a)$ in terms of $u\big|_{\partial\Delta}$.

Remark $a = 0$ Use MVP over the circle $|z| = r$, $r < 1$.

This smaller circle is contained in Δ, where u satisfies MVP.

then

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt$$

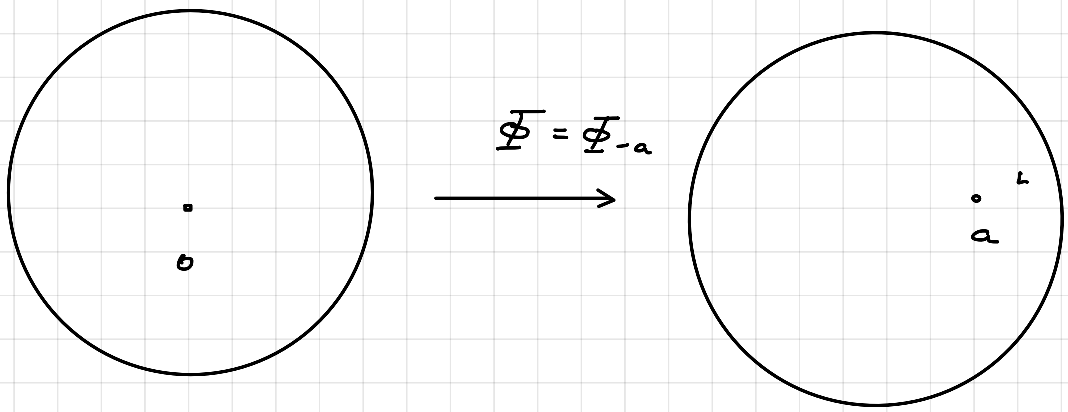
Since u continuous over $\overline{\Delta}$, make $r \longrightarrow 1$. This yields

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\, dt.$$

(To justify the limit use that $u(re^{it}) \longrightarrow u(e^{it})$ uniformly since u is uniformly cont. over $\overline{\Delta}$).

Question : How about the case $a \neq 0$?

General Case



$$\Phi = \Phi_{-a}$$

Idea : Recenter !

$\Phi : \Delta \longrightarrow \Delta, \ \partial\Delta \longrightarrow \partial\Delta$

$$\Phi(z) = \frac{z+a}{1+\bar{a}z}, \quad \Phi(0) = a.$$

Then $\tilde{u} = u \circ \Phi : \bar{\Delta} \longrightarrow \mathbb{R}$ continuous, harmonic in Δ (Problem 1, HWK1)

Apply MVP to $\tilde{u}$

$$u(a) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{is})\, ds = \frac{1}{2\pi}\int_0^{2\pi} u(\Phi(e^{is}))\, ds.$$

Since $\Phi(e^{is}) \in \partial\Delta$ this also shows $u(a)$ is given explicitly in terms of $u|_{\partial\Delta}$.

Next time : We will work out a more explicit expression

$\Longrightarrow$ Poisson Integral Formula

Slogan

MVP + Aut Δ $\Longrightarrow$ Poisson's formula