Math 220C - Lecture 22

May 20, 2022

Let X be a Riemann surface

Let $D = \sum_{p \in X} n_p [p]$ divisor on X.

Question A

Is every divisor D the divisor of a meromorphic function?

Answer depends on X.

[i] non-compact $X \subseteq \mathbb{C}$ open

Yes! Weierstraß Problem

If X compact — additional conditions are needed.

[ii] $X = \hat{\mathbb{C}}$ we need $\deg D = 0$

[iii] $X = \mathbb{C}/\Lambda$ we need $\deg D = 0$ & another condition

What is the issue?

Cover X by coordinate charts U_α with $U_\alpha \overset{\varphi_\alpha}{\cong} G_\alpha \subseteq \mathbb{C}$.

Since we can solve the Weierstraß problem in U_α, we have

$$D\big|_{U_\alpha} = \operatorname{div} f_\alpha \,, \quad f_\alpha \text{ meromorphic in } U_\alpha .$$

Compatibility

$$\operatorname{div} f_\alpha\big|_{U_\alpha \cap U_\beta} = \operatorname{div} f_\beta\big|_{U_\alpha \cap U_\beta} = D\big|_{U_\alpha \cap U_\beta} \Rightarrow$$

$$\Rightarrow \operatorname{div} \frac{f_\alpha}{f_\beta} = 0 \text{ in } U_\alpha \cap U_\beta$$

$\Rightarrow$ f_α / f_β is nowhere zero holomorphic in $U_\alpha \cap U_\beta$.

Want

f meromorphic in X, $\operatorname{div} f = D$

$$\Longleftrightarrow \operatorname{div} f\big|_{U_\alpha} = D\big|_{U_\alpha} = \operatorname{div} f_\alpha$$

$$\Longleftrightarrow \operatorname{div} \frac{f}{f_\alpha} = 0 \text{ in } U_\alpha$$

$\Longleftrightarrow$ f / f_α is nowhere zero holomorphic in U_α

Question A (rephrased) Given

- open cover $X = \bigcup \mathcal{U}_\alpha$, $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$
- $f_\alpha \in \mathcal{M}^*(\mathcal{U}_\alpha)$ with $f_\alpha / f_\beta \in \mathcal{O}^*(\mathcal{U}_{\alpha\beta})$

we want $f \in \mathcal{M}^*(X)$ with

$$f / f_\alpha \in \mathcal{O}^*(\mathcal{U}_\alpha).$$

Notation

- $\mathcal{O}$ = sheaf of holomorphic functions
- $\mathcal{O}^*$ = sheaf of holomorphic nonvanishing fns.
- $\mathcal{M}$ = sheaf of meromorphic functions
- $\mathcal{M}^*$ = nonzero meromorphic functions

Aside : There is a similar additive question.

Question B

Given

$$X = \bigcup \mathcal{U}_\alpha \ , \ f_\alpha \text{ meromorphic in } \mathcal{U}_\alpha$$

such that

$$f_\alpha - f_\beta \in \mathcal{O}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

Want f meromorphic in X with

$$f - f_\alpha \in \mathcal{O}(\mathcal{U}_\alpha).$$

Special case

- $X \subseteq \mathbb{C}$ open, $\mathcal{U}_\alpha$ open near p_α, $p_\beta \notin \mathcal{U}_\alpha$ for $\alpha \neq \beta$

- f_α = Laurent principal part near p_α.

This recovers Mittag-Leffler.

Rephrasing in terms of sheaves

- $\mathcal{M}^*$ = sheaf of meromorphic functions $\neq 0$

- $\underline{\mathcal{D}iv}$ = sheaf of divisors

Recall $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

Rephrasing Question A

Is $H^0(X, \mathcal{M}^*) \longrightarrow H^0(X, \underline{\mathcal{D}iv})$ surjective?

$f \longmapsto \operatorname{div} f$

Strategy for Answering Question A

We rephrase the problem using sheaf cohomology.

Goals

$\boxed{\text{I}}$ **Cohomology**

For all $\mathcal{F} \to X$, we will define $H^p(X, \mathcal{F})$, $p \geq 0$. We already have seen $H^0(X, \mathcal{F})$.

$\boxed{\text{II}}$ **Exact sequences of sheaves**

We will define
- morphisms of sheaves
- exact sequences of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

III Short exact sequences & cohomology

Given $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ we will show

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{H}) \longrightarrow$$
$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{H}) \longrightarrow \cdots$$

We assume this for now & give the details starting next time.

How is this relevant for us?

[a] We will show that

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \xrightarrow{div} \underline{\mathcal{D}iv} \longrightarrow 0 \quad \text{is exact.}$$

[b] If $H^1(X, \mathcal{O}^*) = 0$ $\Rightarrow$ Question A YES

Indeed, $H^0(X, \mathcal{M}^*) \xrightarrow{div} H^0(X, \underline{\mathcal{D}iv}) \longrightarrow \underbrace{H^1(X, \mathcal{O}^*)}_{0}$

$\Rightarrow$ div is surjective

[c] In the additive case, $\mathcal{O}^*$ gets replaced by $\mathcal{O}$.

If $H^1(X, \mathcal{O}) = 0$ $\Rightarrow$ Question B YES

Question C

Given

- $z_1, \dots z_n \in X$, $p_1 \dots p_m \in X$
- $\mu_1, \dots \mu_n \geq 0$, $\nu_1, \dots \nu_m \geq 0$ integers

Want f meromorphic in X

- f has zeroes at z_i of order $\geq \mu_i$
- f has poles at p_i of order $\leq \nu_i$

Other zeroes are allowed, but no other poles.

Let $D = -\sum_i \mu_i [z_i] + \sum_i \nu_i [p_i]$

Want $\operatorname{div} f - \sum_i \mu_i [z_i] + \sum_i \nu_i [p_i] \geq 0$

$\iff$ $D + \operatorname{div} f \geq 0$ (non-negative coefficients).

Sheaves associated to divisors

Given a divisor D, we form a sheaf $\mathcal{O}_X(D)$.

via the assignment

$$U \longrightarrow \{ f \text{ meromorphic} \neq 0,\ \operatorname{div} f + D/_U \geq 0 \} \cup \{0\}.$$

if U connected.

Conclusion Question C is asking to describe

$$V_D = H^0(X, \mathcal{O}_X(D)).$$

Gustav Roch (1839 – 1866)

While in Göttingen, Roch attended lectures of Riemann.

Sheaves on Riemann Surfaces

- $\mathcal{O}$ = sheaf of holomorphic functions
- $\mathcal{O}^*$ = sheaf of holomorphic nonvanishing fns.
- $\mathcal{M}$ = sheaf of meromorphic functions
- $\mathcal{M}^*$ = nonzero meromorphic functions
- $\underline{\mathbb{C}}$ = sheaf of locally constant functions
- $\mathcal{C}^\infty$ = sheaf of smooth functions
- $\underline{\mathcal{D}iv}$ = sheaf of divisors
- $\mathcal{O}_X(D)$ = sheaf associated to divisor D.

$\boxed{\text{I}}$ These sheaves solve different problems

$\boxed{\text{II}}$ These sheaves interact with each other