Math 220C - Lecture 21

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Homological methods & sheaves

$\boxed{\text{I}}$ define morphisms

$\boxed{\text{II}}$ define exact sequences

Morphisms of sheaves

$\mathcal{F}, \mathcal{G} \to X$ sheaves on a topological space

A morphism of sheaves

$\alpha : \mathcal{F} \to \mathcal{G}$ consists in homomorphisms

$\alpha_u : \mathcal{F}(u) \to \mathcal{G}(u)$ $\forall\ u \subseteq X$ open.

We require that

$$\begin{tikzcd} \mathcal{F}(u) \arrow[r, "\alpha_u"] \arrow[d, "\rho^{\mathcal{F}}_{uv}"'] & \mathcal{G}(u) \arrow[d, "\rho^{\mathcal{G}}_{uv}"] \\ \mathcal{F}(v) \arrow[r, "\alpha_v"] & \mathcal{G}(v) \end{tikzcd}$$

$\forall\ v \subseteq u$ open

Remark Given $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ we obtain $\forall\, x \in X$

$$\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

Why? Let $f_x \in \mathcal{F}_x$. Represent f_x by (f, U), $x \in U$, $f \in \mathcal{F}(U)$. Define

$$\alpha_x(f_x) = \alpha(f)_x = \text{germ of } \alpha_U(f) \text{ at } x.$$

Since α is compatible with restrictions, the definition is independent of choices.

Exact sequences

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \quad \text{is exact iff}$$

$$\forall\, x \in X, \quad 0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0 \quad \text{is exact.}$$

Lemma If $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$ exact then

$\forall\, U \subseteq X$, open $\quad 0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ exact.

Proof WLOG $U = X$. Else work with the sheaves

$\mathcal{F}/_U$, $\mathcal{G}/_U$, $\mathcal{H}/_U$ noting that

$$0 \longrightarrow \mathcal{F}/_U \longrightarrow \mathcal{G}/_U \longrightarrow \mathcal{H}/_U \longrightarrow 0$$

since the stalks at $x \in U$ do not change by restriction.

Remark $f = 0 \iff f_x = 0$ for f section of a sheaf $\mathcal{F}$

Proof "$\Leftarrow$": Since $f_x = 0 \overset{def}{\Longrightarrow} f = 0$ in $W_x \ni x$. open.

Since $X = \bigcup_x W_x$, it follows $f = 0$ in X by

uniqueness of gluing.

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(1) $\alpha : \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ injective.

Assume $\alpha(f) = 0$. For $x \in X \Rightarrow \alpha(f)_x = 0 \Rightarrow$

$\Rightarrow \alpha_x(f_x) = 0$

α_x injective

$\Rightarrow f_x = 0 \overset{\text{Remark}}{\Rightarrow} f = 0$.

(2) $\beta \circ \alpha = 0$ over X

Let $f \in \mathcal{F}(X)$. Note

$$(\beta \circ \alpha)(f)_x = \beta_x \alpha_x(f_x) = 0 \quad \text{since}$$

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0 \quad \text{is exact.}$$

By the Remark we see $(\beta \circ \alpha)(f) = 0 \Rightarrow \beta \circ \alpha = 0$.

(3) $\mathrm{Ker}\,\beta_X \subseteq \mathrm{Im}\,\alpha_X$

Let $g \in \mathcal{G}(X)$, $\beta(g) = 0$. Then $\beta_x(g_x) = 0 \quad \forall x \in X$

$\Rightarrow \exists f_x \in \mathcal{F}_x$ with $g_x = \alpha_x(f_x)$ by exactness of

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0$$

Represent the germ f_x by a section (f^x, U^x) with

$$g = \alpha(f^x) \text{ in } U^x.$$

Note $\alpha(f^x|_{U^x \cap U^y}) = \alpha(f^y|_{U^x \cap U^y}) = g|_{U^x \cap U^y}$. We

proved α is injective in Step (1) so

$$f^x|_{U^x \cap U^y} = f^y|_{U^x \cap U^y}.$$

By gluing, we can find $f \in \mathcal{F}(X)$ with

$$f|_{U^x} = f^x.$$

Then $\alpha(f)|_{U^x} = \alpha(f^x) = g|_{U^x} \Rightarrow \alpha(f) = \beta$ by

sheaf axioms. This is what we needed.

Remark Assume we are given

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \quad \text{such that}$$

$\forall\, U \subseteq X$ open $\quad 0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact.

Then $\quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact.

Why We argue $0 \longrightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \longrightarrow 0$ exact.

We need to show

$\mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is surjective. The rest is covered by the arguments above.

Take $h_x \in \mathcal{H}_x$, represent it by (h, U). Write

$h = \beta(g)$ since $\beta : \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$ surjective.

Then $h_x = \beta_x(g_x)$ with $g_x \in \mathcal{G}_x$, as needed.

Conclusion

(1) $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact $\Longrightarrow$

$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$ exact $\forall\, U \subseteq X$ open

Exactness on the right may fail.

(2) If $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact

for a basis of neighborhoods $\{U\}$ in X $\Longrightarrow$

$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$ exact.

This follows from the argument on previous page

Three Examples - Exponential sequence

Let X be a Riemann surface.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathcal{O} \xrightarrow{\beta} \mathcal{O}^* \longrightarrow 1 \quad \text{exact}$$

The morphisms α and β

$\alpha(1) = 1$, $\beta(f) = e^{2\pi i f}$.

Why exact β is surjective on a basis consisting of simply connected coordinate charts. This follows since $\log$'s of nowhere zero functions are defined by Math 220 A.

β is not surjective on global sections

$X = \mathbb{C}^\times$, $\beta : f \longrightarrow e^{2\pi i f}$. Note $\operatorname{Im} \beta_X$ does not contain the function z since $\log z$ is not defined in $\mathbb{C}^\times$.

$\Rightarrow$ $\mathcal{O}(X) \longrightarrow \mathcal{O}^*(X)$ not surjective.

Example

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{\alpha} \mathcal{M}^* \xrightarrow{\beta} \underline{\mathcal{D}iv} \longrightarrow 0 \quad \text{exact.}$$

The morphisms α and β

$\alpha(f) = f$

$\Longrightarrow \beta \circ \alpha = 0$

$\beta(g) = \operatorname{div} g$

Why exact

We check β is surjective on a basis consisting of coordinate charts. By Weierstraß Problem, in such a chart, every divisor is the divisor of a meromorphic function proving surjectivity of $\beta : \mathcal{M}^* \longrightarrow \underline{\mathcal{D}iv}$.

Example Let $D = \sum n_j [p_j]$, $n_j \geq 0$. Then

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X(D) \xrightarrow{\beta} \prod \underbrace{\mathbb{C}_{p_j}}^{\oplus n_j} \longrightarrow 0 \quad \text{exact}$$

skyscraper sheaf of p_j

The morphisms α and β

- $\alpha(f) = f \implies \operatorname{div} f + D \geq 0$ since $D \geq 0$, $\operatorname{div} f \geq 0$

$\implies \alpha$ is well-defined

- $\beta(f) = \prod_j \left(c_{-n_j}^{(p_j)}, \dots, c_{-1}^{(p_j)}\right) \in \prod_j \mathbb{C}_{p_j}^{\oplus n_j}$

where the c's are the Laurent coefficients of f near p_j:

$$f = \frac{c_{-n_j}}{(z-p_j)^{n_j}} + \dots + \frac{c_{-1}}{z-p_j} + \dots$$

Why exact

β is surjective on a basis consisting of coordinate charts. by Mittag-Leffler in open subsets of $\mathbb{C}$.