Math 2200 - Jeoture 21

May 23 , 2022

Homological methods & sheaves

1) define morphisms

Indefine exact sequences

Morphisms of sheaves

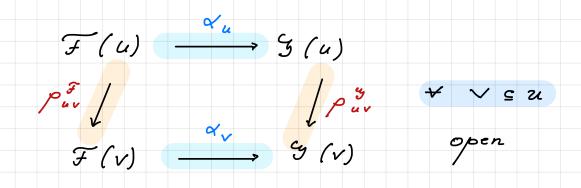
F. 4 . X sheaves on a topological space

A morphism of sheaves

a: F-+g consists in homomorphisms

 α_{u} : $\mathcal{F}(u) \longrightarrow \mathcal{G}(u) + u \subseteq \times open.$

We require that



Remark Given a: F-y we obtain + * E ×

 $\alpha_{n}: \mathcal{F} \longrightarrow \mathcal{Y}_{n}$

Why? Let fre Fr. Represent fr by (f, u), reu,

f & F (u). Define

 $\alpha_{x}(f_{x}) = \alpha(f)_{x} = germ of \alpha(f) af x.$

Since & is compatible with restrictions, the definition

is independent of choices.

Exact seguences

 $o \longrightarrow \mathcal{F} \xrightarrow{a} \mathcal{C}_{\mathcal{G}} \xrightarrow{\mathcal{B}} \mathcal{J}_{\mathcal{H}} \longrightarrow o \quad is \quad exact \quad if f$

 $\forall x \in X, 0 \longrightarrow \overline{f_x} \xrightarrow{d_x} G_x \xrightarrow{\beta_x} \overline{f_y} \longrightarrow \overline{f_y} \longrightarrow 0$ is exact.

Temma If 0 -> F -> y B. J-1 -> 0 rexact then

 $* \ u \subseteq X, \ open \ o \longrightarrow \mathcal{F}(u) \xrightarrow{d_u} \mathcal{G}(u) \xrightarrow{\mathcal{B}_u} \mathcal{F}(u) \xrightarrow{e_{xact}}$

Proof WLOG U=x. Else work with the sheaves

F/u, G/u, Fl/u noting that

 $o \longrightarrow \mathcal{F}/_{u} \longrightarrow \mathcal{G}/_{u} \longrightarrow \mathcal{H}/_{u} \longrightarrow o$

Since the stalks at x & U do not change by restriction.

Remark f=0 <=> f= o for f section of a sheaf F

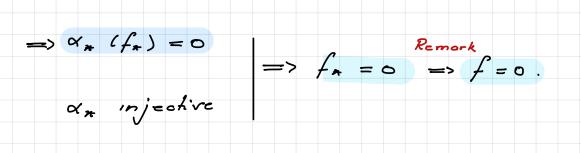
 $\frac{Proof}{m} \stackrel{"}{\leqslant} = \stackrel{"}{\underset{n}{\overset{\circ}{\sim}}} 5ince f_* = 0 \implies f = 0 \text{ in } W_* \ni *. \text{ open.}$

Since $X = U W_{x}$, it follows f = 0 in X by

unique meso of gluing

(1) α : $\overline{f}(x) \longrightarrow \mathcal{G}(x)$ in jechive.

Assume $\alpha(f) = 0$. For $\pi \in X = \lambda \alpha(f)_{\pi} = 0 = \lambda$





Let fe F(x). Nok

 $(\beta \cdot \alpha)(f)_{\mu} = \beta_{\mu} \alpha_{\mu}(f_{\mu}) = 0$ since

 $\longrightarrow \quad \overrightarrow{\mathcal{F}_{x}} \xrightarrow{\alpha_{x}} \overset{q_{x}}{\longrightarrow} \overset{f^{3}x}{\longrightarrow} \overrightarrow{\mathcal{H}}_{x} \longrightarrow o \quad is \quad exact.$

By the Remark WE see (Boa) (f) = 0 => Boa = 0.

(3) Ker Bx Elmax

Jet ge G(x), B(g)=0. Then B, (g,)=0 + x ex

=> $\mathcal{F}_{f_{\mathbf{x}}} \in \mathcal{F}_{\mathbf{x}}$ with $g_{\mathbf{x}} = \alpha_{\mathbf{x}}(f_{\mathbf{x}})$ by exactness of

Represent the germ fx by a section (f, u*) with

 $g = \alpha (f^*)$ in u^*

Note $\alpha \left(f^{*} / u^{*} n u^{y} \right) = \alpha \left(f^{*} / u^{*} n u^{y} \right) = g / u^{*} n u^{y}$. We

proved a is injective in Step (1) so

 $f^*/u^*nu^y = f^y/u^*nu^y$

By gluing, we can find f E F(x) with

 $f/u^* = f^*$

Then $\alpha(f)|_{u^*} = \alpha(f^*) = g|_{u^*} = \alpha(f) = \beta b_f$

sheaf axioms. This is what we needed.

Remark Assume we are given

 $o \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{F} \mathcal{I} \longrightarrow o \quad such that$

* u = x open 0 - F(u) - G(u) - H(u) - o react.

Then $o \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow o$ exact.

Why We argue 0 - Fr dr G Br Ha - o exact.

We need to show

Gy 13* Hy is surjective. The rest is covered by

the arguments above.

Take h & E H * , represent it by (h, u). Write

h = p(g) since p: g(u) - F/(u) surjective.

Then $h_* = \beta_* (g_*)$ with $g_* \in \mathcal{G}_*$, as needed.

Conclusion

(1) $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow$

0 - F(us - G(u) - F(u) exact + u = x open

Exactness on the right may fail.

(2) If 0 - F(u) - G(u) - F(u) - o exact

for a basis of meighbor hoods fug in X =>

 $o \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow o$. exact.

This follows from the argument on previous page

Three Examples - Exponential sequence

Jet X be a Riemann surface.

 $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{O} \xrightarrow{\beta} \mathbb{O}^* \longrightarrow \mathbb{I} = x a o t$

The morphisms & and B_

 $\alpha(1) = 1$, $\beta(f) = e$.

Why exact B is surjective on a basis consisting of

simply connected coordinate charts. This follows since log's

of nowhere zero functions are defined by Moth 220 A.

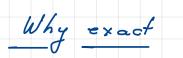
B is not surjective on global sections $X = C^{X}$, $\beta : f \longrightarrow c^{2\pi i f}$. Note Im β_{X} does not contain the function 2 since log 2 is not defined in C.

=> $O(x) \rightarrow G^{*}(x)$ not surjective.

Example

 $o \longrightarrow G^* \xrightarrow{\alpha} \mathcal{M}^* \xrightarrow{\beta} \mathcal{D}_i v \longrightarrow o = xact.$

The morphisms & and B_ α (f) = f => /3 • ~ = 0 $\beta(g) = div g$



We check B is surjective on a basis consisting of

coordinate charts. By Weizzshaps Problem, in such a chart,

every divisor is the divisor of a meromorphic function

proving surjectivity of p: M* - Div.

Example $Zet D = \sum n_j [p_j, 7, n_j; zo. Then$

 $0 \longrightarrow \mathcal{O}_{\mathsf{x}} \xrightarrow{\alpha} \mathcal{O}_{\mathsf{x}} (\mathbf{o}) \xrightarrow{\beta} 77 \xrightarrow{\mathcal{O}} \mathbf{f}_{p_{j}}^{\mathcal{O}} \xrightarrow{\mathbf{o}} 0 \quad \mathbf{e} \times \mathbf{o} \mathbf{c} \mathbf{f}$

skyscraper sheaf of pj.

The morphisms α and β . • $\alpha(f) = f$ => div $f + D \ge 0$ since $D \ge 0$, div $f \ge 0$

=) a is well-defined

• $f(f) = \frac{77}{j} \left(\begin{array}{c} c \\ -n_{j} \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \\ -n_{j} \end{array}, \begin{array}{c} c \\ -n_{j} \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \\ -n_{j} \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \\ -n_{j} \end{array}, \begin{array}{c} c \\ -n_{j} \end{array}, \begin{array}{c} c \end{array}, \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \end{array}, \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \end{array}, \begin{array}{c} c \end{array}, \end{array}, \end{array}, \begin{array}{c} c \end{array}, \end{array}, \end{array}, \end{array}, \begin{array}{c} c \end{array}, \end{array}, \end{array},$

where the c's are the Laurent coefficients of f near p;

 $f = \frac{C_{-nj}}{(2-p_j)^{n_j}} + \cdots + \frac{C_{-l}}{2-p_j} + \cdots$

Why exact

B is surgecture on a basis consisting of coordinate

charts. by Mittag - Jeffler in open subsets of a.