Math 220C - Lecture 22

May 24, 2022

§ 1. Homological Methods & Sheaves (Part II)

Given $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves, we define

[i] the sheaf $\operatorname{Ker} \alpha$

[ii] the sheaf $\operatorname{Coker} \alpha$

Kernel When $U \subseteq X$ open, set

$$\operatorname{Ker} \alpha\,(U) = \operatorname{Ker} \{ \alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \}$$

Restriction maps

$\operatorname{Ker} \alpha\,(U) \longrightarrow \operatorname{Ker} \alpha\,(V)$ are naturally defined.

Check [i] $\operatorname{Ker} \alpha$ is a sheaf

[ii] $(\operatorname{Ker} \alpha)_x = \operatorname{Ker} \{ \alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x \}.$

Cokernel Presheaf

For $U \subseteq X$ open, define

$$\widetilde{\mathrm{Coker}\,\alpha}\,(U) = \mathrm{Coker}\left\{ \alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \right\}$$

Check

$\boxed{\text{I}}$ $\widetilde{\mathrm{Coker}\,\alpha}$ is a presheaf

$\boxed{\text{II}}$ $\left(\widetilde{\mathrm{Coker}\,\alpha}\right)_x = \mathrm{Coker}\,(\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x)$

Beware! $\widetilde{\mathrm{Coker}\,\alpha}$ is not always a sheaf

Example $X = \mathbb{C}^\times$, $\mathcal{O} \xrightarrow{\alpha} \mathcal{O}^*$, $\alpha(f) = e^{2\pi i f}$

Let $\mathcal{F} = \widetilde{\mathrm{Coker}\,\alpha}$

- $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $\mathcal{F}(U) = 0$

Indeed, U is simply connected, so logarithms make sense $\Rightarrow$ α is surjective in U $\Rightarrow$ $\widetilde{\mathrm{Coker}}\,\alpha_U = 0$.

- $V = \mathbb{C} \setminus \mathbb{R}_{\geq 0} \Rightarrow \mathcal{F}(V) = 0$

- $X = U \cup V = \mathbb{C}^\times$. Note α is not surjective in X since $\log z$ is not defined in $\mathbb{C}^\times$. $\Rightarrow$ $\mathcal{F}(X) \neq 0$.

Any $s \in \mathcal{F}(X)$ restricts $s|_U = 0$, $s|_V = 0$ since $\mathcal{F}(U) = 0$, $\mathcal{F}(V) = 0$. If $s \neq 0$ this contradicts uniqueness of gluing $\Rightarrow$ $\mathcal{F}$ not a sheaf.

Sheafification

Goal

Given $\mathcal{F} \longrightarrow X$ a presheaf, we define a sheaf $\mathcal{F}^{\#}$

& morphism of presheaves

$$i : \mathcal{F} \longrightarrow \mathcal{F}^{\#}.$$

Remark In addition,

[i] $\mathcal{F}$ sheaf $\Rightarrow$ $\mathcal{F} = \mathcal{F}^{\#}$

[ii] $\mathcal{F}_x = \mathcal{F}^{\#}_x \quad \forall x$

[iii] Given $\mathcal{F} \longrightarrow \mathcal{G}$ we obtain $\mathcal{F}^{\#} \longrightarrow \mathcal{G}^{\#}$ with

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F}^{\#} \arrow[r] & \mathcal{G}^{\#} \end{tikzcd}$$

commutative.

Definition

$$\mathcal{F}^{\#}(U) = \Big\{ (f_x)_{x\in U} \in \mathcal{F}_x, \text{ "locally compatible germs" i.e.}$$

$$\forall\, x \in U \quad \exists\, x \in V \subseteq U,\ s \in \mathcal{F}(V) \ \text{ with } s_y = f_y \ \ \forall\, y \in V \Big\}$$

Example

$\mathcal{F}$ = presheaf of constant functions

$\mathcal{F}^{\#}$ = sheaf of locally constant functions

Remark

We define $\mathcal{F} \longrightarrow \mathcal{F}^{\#}$ via

$$\mathcal{F}(U) \ni f \longrightarrow (f_x)_x \in \mathcal{F}^{\#}(U).$$

Check

$\mathcal{F}^{\#}$ is a sheaf & $\mathcal{F}_x = \mathcal{F}^{\#}_x$.

Conclusion — Cokernel sheaf

Given $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$, we define the cokernel sheaf:

(1) $\widetilde{\operatorname{Coker}\alpha}$ presheaf

(2) sheafify $\operatorname{Coker}\alpha := \widetilde{\operatorname{Coker}\alpha}^{\#}$.

Why does it work?

Assume $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$. We have by definition

$$\mathcal{G} \longrightarrow \widetilde{\operatorname{Coker}\alpha}$$

This gives

$$\mathcal{G} = \mathcal{G}^{\#} \longrightarrow \widetilde{\operatorname{Coker}\alpha}^{\#}.$$

Then

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \widetilde{\operatorname{Coker}\alpha}^{\#} \longrightarrow 0 \quad \text{exact}$$

as needed.

Exactness can be checked on stalks. We note that

$$\left(\widetilde{\operatorname{Coker}\alpha}^{\#}\right)_x = \left(\widetilde{\operatorname{Coker}\alpha}\right)_x$$

$$= \operatorname{Coker}(\alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x).$$

$$\Rightarrow \quad 0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \left(\widetilde{\operatorname{Coker}\alpha}^{\#}\right)_x \longrightarrow 0$$

exact, as needed.

§ 2. Flabby sheaves

$\mathcal{F}$ is *flabby* provided $\forall\ V \subseteq U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \text{ is surjective.}$$

Remarks $\boxed{\text{i}}$ $\mathcal{F}$ flabby $\Rightarrow$ $\mathcal{F}|_U$ flabby $\forall\ U \subseteq X$ open

Indeed, for $V \subseteq W \subseteq U$,

$$\mathcal{F}|_U(W) = \mathcal{F}(W) \longrightarrow \mathcal{F}(V) = \mathcal{F}|_U(V) \text{ surjective.}$$

$\boxed{\text{ii}}$ Suffices to check $\forall\ U \subseteq X$ open

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \text{ surjective}$$

Indeed,

$$\begin{tikzcd} \mathcal{F}(X) \arrow[rr] \arrow[dr] & & \mathcal{F}(U) \\ & \mathcal{F}(V) \arrow[ur] & \end{tikzcd}$$

shows $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ also surjective for $U \subseteq V$.

Key Lemma

If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ exact, $\mathcal{F}$ flabby then

$\boxed{i}$ $0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$ exact, $\forall\, U \subseteq X$ open

$\boxed{ii}$ $\mathcal{F}, \mathcal{G}$ flabby $\Longrightarrow$ $\mathcal{H}$ flabby

Proof $\boxed{i} \Longrightarrow \boxed{ii}$

Let $V \subseteq U$ open. Compare the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, "\text{surj}"] & \mathcal{G}(U) \arrow[r] \arrow[d, "\text{surj}"] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{H}(U) \arrow[r] \arrow[d, "\text{surj}"] & 0 \\
0 \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \arrow[r] & \mathcal{H}(V) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \mathcal{H}$ flabby.